

TATE OBJECTS IN EXACT CATEGORIES

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ABSTRACT. We study elementary Tate objects in an exact category. We characterize the category of elementary Tate objects as the smallest sub-category of admissible Ind-Pro objects which contains the categories of admissible Ind-objects and admissible Pro-objects, and which is closed under extensions. We compare Beilinson's approach to Tate modules to Drinfeld's. We establish several properties of the Sato Grassmannian of an elementary Tate object in an idempotent complete exact category (e.g. it is a directed poset). We conclude with a brief treatment of n -Tate modules and n -dimensional adèles.

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1. INTRODUCTION

In the article *Residues of Differentials on Curves* [29] J. Tate developed a new understanding of the classical theory of residues. Given a differential fdg on a curve X defined over a field k , the residue at a point x can be defined as the trace of a suitable operator assigned to f and g acting on the infinite-dimensional vector space $\widehat{F}_x \cong k_x((t))$. This unprecedented viewpoint on residues immediately implies the independence of local coordinates and allowed Tate to give an intrinsic proof of the *sum-of-residues theorem*. The work of Parshin [20], Arbarello–de Concini–Kac [1] and Beilinson [3] brought Tate’s techniques to new heights.

A conceptual approach to the infinite-dimensional vector space of formal Laurent series $k((t))$ is provided by Lefschetz’s theory of *locally linearly compact* vector spaces [16, Chapter II.6]. A topological vector space U over a discrete field k is said to be *discrete* if it has the discrete topology. The *topological dual* U^\vee of a discrete vector space is called a *linearly compact* vector space. A *locally linearly compact* vector space W can be written as an extension of a discrete vector space V

$$0 \longrightarrow U^\vee \longrightarrow W \longrightarrow V \longrightarrow 0$$

by a *linearly compact* vector space U^\vee .

For the example of formal Laurent series one endows $k((t))$ with the finest linear topology such that $t^n \rightarrow 0$ for $n \rightarrow \infty$. The aforementioned extension is induced by the direct sum decomposition $k((t)) = k[[t]] \oplus k((t))/k[[t]]$.

While the theory of locally linearly compact vector spaces may be sufficient for the purpose of (equal characteristic) algebraic geometry, arithmetic considerations necessitate an analogous treatment of *locally linearly compact* abelian groups. Indeed, the short exact sequence

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{Q}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0$$

certainly realizes the p -adic numbers \mathbb{Q}_p as an extension of the *compact* abelian group \mathbb{Z}_p by a discrete group.

Moreover, Parshin and Beilinson’s theory of adèles ([20],[3]) and Drinfeld’s theory of infinite-dimensional vector bundles [10] suggest that the constructions above should be *iterated* and *studied in families*. The first point is important to the study of higher-dimensional varieties. The second point concerns the transition from k -vector spaces to R -modules, which allows us to consider various moduli problems.

In this article we develop the theory of elementary Tate objects in an exact category \mathcal{C} . We begin by introducing the exact category $\mathrm{Ind}^a(\mathcal{C})$ of *admissible Ind-objects*, which is constructed from \mathcal{C} by adding certain formal colimits. We also introduce the category $\mathrm{Pro}^a(\mathcal{C})$ of *admissible Pro-objects*, which is constructed from \mathcal{C} by adding certain formal limits. This allows us to introduce the category $\mathrm{Ind}^a(\mathrm{Pro}^a(\mathcal{C}))$ of *admissible Ind-Pro objects*, and to define the category $\mathrm{Tate}^{el}(\mathcal{C})$ of elementary Tate objects as a full sub-category of $\mathrm{Ind}^a\mathrm{Pro}^a(\mathcal{C})$. Our first main result is the following characterization of $\mathrm{Tate}^{el}(\mathcal{C})$.

Theorem 1.1. *5.4 Let \mathcal{C} be an exact category. The category $\mathbf{Tate}^{el}(\mathcal{C})$ of elementary Tate objects is the smallest full sub-category of $\mathbf{Ind}^a(\mathbf{Pro}^a(\mathcal{C}))$ which*

- (1) *contains the sub-category $\mathbf{Ind}^a(\mathcal{C}) \subset \mathbf{Ind}^a(\mathbf{Pro}^a(\mathcal{C}))$,*
- (2) *contains the sub-category $\mathbf{Pro}^a(\mathcal{C}) \subset \mathbf{Ind}^a(\mathbf{Pro}^a(\mathcal{C}))$, and*
- (3) *is closed under extensions.*

The categories $\mathbf{Ind}^a(\mathcal{C})$, $\mathbf{Pro}^a(\mathcal{C})$ and $\mathbf{Tate}^{el}(\mathcal{C})$ play a great role in various applications, notably in algebraic K -theory (e.g. [27], [24], [8]), in Drinfeld’s study of infinite-dimensional vector bundles in algebraic geometry [10], in Parshin and Beilinson’s theory of multidimensional adèles of schemes ([20],[3]), in reciprocity laws (e.g. [19],[18],[7]), and in de Rham epsilon factors [5].

This theorem provides a basic tool for producing elementary Tate objects. It also indicates how the theory of elementary Tate objects should be generalized to homological settings. Indeed, there is no difficulty in formulating the analogues of admissible Ind and Pro-objects for DG-categories and stable ∞ -categories, and we can then take the theorem as the definition of an elementary Tate object. While we do not pursue this here, this theory will allow for applications to the study of perfect complexes and the algebraic K -theory of schemes.

The notion of *Tate objects* in an arbitrary exact category was introduced by Beilinson [4] and has also been recently studied by Previdi in [21]; if we restrict to countable indexing diagrams, our “elementary Tate objects” coincide with their approach (Proposition 5.16). Moving beyond countable diagrams allows us to treat examples such as the adèles of curves over fields of uncountable cardinality. A related notion of Tate modules has been recently introduced by Drinfeld [10].¹ We show that for countably generated modules, Drinfeld’s category is the idempotent completion of Beilinson’s, and that for uncountably generated modules, Drinfeld’s category is a full sub-category of the idempotent completion of $\mathbf{Tate}^{el}(\mathcal{C})$ (Theorem 5.22). Motivated by this and other examples of algebraic geometry, we study the categories $\mathbf{Ind}_\kappa^a(\mathcal{C})$, $\mathbf{Pro}_\kappa^a(\mathcal{C})$, and $\mathbf{Tate}_\kappa^{el}(\mathcal{C})$ in light of *idempotent completeness*; none of these categories inherits *idempotent completeness* from \mathcal{C} (see Section 3.2.6).

We next consider *lattices* in an elementary Tate object V . The lattice $k[[t]]$ in the elementary Tate vector space $k((t))$ provides a canonical example. We study the *Sato Grassmannian* of lattices in an elementary Tate object. Sato and Sato introduced these Grassmannians in their study of integrable hierarchies of differential equations (e.g. [25]). A key feature of the theory is that if V is an elementary Tate object in \mathcal{C} , then the quotient of a lattice by a sub-lattice is an object in \mathcal{C} . A second key feature, exploited by Sato and Sato, is that given any two lattices L_0 and L_1 in an elementary Tate vector space, there exists a common sub-lattice N contained in L_0 and L_1 , and a common enveloping lattice containing them both. We show that the first feature holds in general (Proposition 6.5). Our second main result is the following.

Theorem 1.2. *6.6 Let \mathcal{C} be an idempotent complete exact category. The poset underlying the Sato Grassmannian $Gr(V)$ of an elementary Tate object V in \mathcal{C} is directed.*

The existence of common sub-lattices and common enveloping lattices turns out to be key for applications (e.g. in [8], [7]). Moreover, the assumption of idempotent

¹As an unfortunate consequence of Tate’s mathematical creativity, there are also several other and very different notions of “Tate module”.

completeness is satisfied in many applications, for instance if \mathcal{C} is the category of vector bundles on a scheme.

Sato Grassmannians for elementary Tate objects in exact categories \mathcal{C} have been studied recently by Previdi [22]. In order to ensure their good behavior, Previdi introduced two properties for exact categories: “partially abelian” and “AIC plus AIC^{op} ”. The first notion turns out to be unnecessarily strong and, as was explained to us by T. Bühler, is equivalent to being abelian. The second condition, according to Bühler, is equivalent to Schneider’s notion of “quasi-abelian” [28] and Rump’s “almost abelian” [23].

Unfortunately, many exact categories of interest fail to satisfy “AIC plus AIC^{op} ”. A basic example is the category of vector bundles over the real line \mathbb{R} : the intersection of the admissible monics

$$\begin{aligned} \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R} \times \mathbb{R}^2 \\ (x, t) &\longmapsto (x, (t, 0)) \end{aligned}$$

and

$$\begin{aligned} \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R} \times \mathbb{R}^2 \\ (x, t) &\longmapsto (x, (t, xt)) \end{aligned}$$

is not a vector bundle.

We replace this condition with idempotent completeness, in part to handle geometric examples such as the above.

We conclude this article with a brief discussion of *n-Tate objects*. As a motivating example, we show that the *n*-dimensional adèles of an *n*-dimensional Noetherian scheme are canonically an *n*-Tate object (for a closely related result, see [17]). The structure of the Sato Grassmannian of the adèles, or more generally of an arbitrary *n*-Tate object, is the subject of continuing investigation.

Outline. In Sections 3 and 4, we develop the basic theory of admissible Ind-objects and admissible Pro-objects. Our treatment closely follows [14, Appendix B], though in considering uncountable cardinalities we need to slightly adapt Keller’s methods. This material should be understood as an elaboration, for exact categories, of the appendix to [2]. In Section 5, we introduce elementary Tate objects and establish Theorem 5.4. Many of the results in this section should be familiar to experts, but we could not find them in the literature except for countable elementary Tate objects, as in [21]. In Section 6, we consider lattices and Sato Grassmannians. In Section 7, we introduce the category of *n*-Tate objects and summarize its properties. We show that the Beilinson–Parshin adèles of an *n*-dimensional Noetherian scheme are *n*-Tate objects. Appendix A repeats a proof due to T. Bühler which shows that the present approach to left s-filtering sub-categories is equivalent to Schlichting’s.

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2. PRELIMINARIES

2.1. Cardinal Arithmetic. We recall the following standard lemma.

Lemma 2.1. *Let κ be an infinite cardinal. Let K be a set of cardinality κ . The disjoint union $\coprod_{n \in \mathbb{N}} K^n$ of all finite tuples of elements of K has cardinality κ . In particular, the disjoint union of all finite subsets of K has cardinality at most κ .*

Proof. The proof is a standard induction, using the facts that $|K \times K| = \kappa$ and $|K \coprod K| = \kappa$. \square

2.2. Exact Categories. Exact categories provide a general framework for linear algebra. We refer the reader to Bühler's survey [9] for a full treatment.

Definition 2.2. Let \mathcal{C} be an additive category. A *kernel-cokernel pair* is a sequence

$$X \hookrightarrow Y \twoheadrightarrow Z$$

such that $X \hookrightarrow Y$ is the kernel of $Y \twoheadrightarrow Z$, and $Y \twoheadrightarrow Z$ is the cokernel of $X \hookrightarrow Y$.

An *exact category* is an additive category \mathcal{C} equipped with a class \mathcal{EC} of kernel-cokernel pairs. An *exact sequence* is a kernel-cokernel pair in \mathcal{EC} . An *admissible monic* is a map $X \hookrightarrow Y$ which serves as the kernel in an exact sequence; an *admissible epic* is $Y \twoheadrightarrow Z$ is a map which serves as a cokernel in an exact sequence. We require that

- (1) for all $X \in \mathcal{C}$, the identity 1_X is both an admissible monic and an admissible epic,
- (2) the class of admissible monics and the class of admissible epics are closed under composition,
- (3) the pushout of an admissible monic along an arbitrary morphism exists and is an admissible monic, and the pullback of an admissible epic along an arbitrary morphism exists and is an admissible epic.

A functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ between exact categories is *exact* if $F(\mathcal{EC}) \subset \mathcal{ED}$. A fully faithful embedding $F: \mathcal{C} \longrightarrow \mathcal{D}$ is *fully exact* if every exact sequence in \mathcal{D} of the form $F(X) \hookrightarrow F(Y) \twoheadrightarrow F(Z)$ is the image of an exact sequence in \mathcal{C} .

Every additive category \mathcal{C} defines an exact category where the class \mathcal{EC} consists of all split exact sequences. The category $P^f(R)$ of finitely generated projective modules over a ring R provides a motivating example. For another source of examples, every abelian category \mathcal{C} defines an exact category in which \mathcal{EC} is the class of all kernel-cokernel pairs.

2.2.1. Idempotent Completeness. We recall two conditions on exact categories: idempotent completeness and weak idempotent completeness. In practice, the former is both more important and better behaved than the latter. The category $F^f(R)$ of finitely generated free modules over a ring R provides an example of an exact category which is not idempotent complete. The category $P^f(R)$ of finitely generated projective R -modules provides an example of one which is.

Definition 2.3. An exact category \mathcal{C} is *weakly idempotent complete* if every retract has a kernel. Explicitly, we require that any map $r: X \rightarrow Y$ for which there exists a right inverse $s: Y \rightarrow X$ admits a kernel in \mathcal{C} .

Remark 2.4. This condition is actually self-dual. For any additive category \mathcal{C} , all retracts have kernels in \mathcal{C} if and only if all retracts have kernels in \mathcal{C}^{op} . See [9, Lemma 7.1].

Definition 2.5. An exact category \mathcal{C} is *idempotent complete* if, for every $p: X \rightarrow X$ such that $p^2 = p$, there exists an isomorphism $X \cong Y \oplus Z$ which takes p to the projection $0 \oplus 1_Z$.

Example 2.6. [9, Exercise 7.11] Let $R = \mathbb{Q} \times \mathbb{Q}$. The category of finitely generated free R -modules is weakly idempotent complete, but not idempotent complete.²

Definition 2.7. Let \mathcal{C} be a category. Define the *idempotent completion* \mathcal{C}^{ic} of \mathcal{C} to be the category whose objects are pairs (X, p) , with $p: X \rightarrow X$ an idempotent in \mathcal{C} . Morphisms $(X, p) \rightarrow (Y, q)$ in \mathcal{C}^{ic} correspond to morphisms $g: X \rightarrow Y$ in \mathcal{C} such that $qgp = g$; composition is induced by composition in \mathcal{C} .

Example 2.8. Let $F^f(R)$ and $P^f(R)$ denote the categories of finitely generated free and projective R -modules. The idempotent completion $F^f(R)^{ic}$ is equivalent to $P^f(R)$.

The assignment $X \mapsto (X, 1)$ defines a fully faithful embedding $\mathcal{C} \hookrightarrow \mathcal{C}^{ic}$. We do not distinguish between \mathcal{C} and its essential image under this embedding.

Proposition 2.9. [9, Proposition 6.10] *Let \mathcal{C} be an exact category. Let \mathcal{C}^{ic} be the idempotent completion of \mathcal{C} . Define $\mathcal{E}(\mathcal{C}^{ic})$ to consist of sequences which are direct summands of exact sequences in \mathcal{C} . Explicitly, a sequence*

$$(X_0, p_1) \rightarrow (X_1, p_1) \rightarrow (X_2, p_2)$$

is exact in \mathcal{C}^{ic} if there exists a sequence

$$(X'_0, p'_0) \rightarrow (X'_1, p'_1) \rightarrow (X'_2, p'_2)$$

such that, for all i , $(X_i, p_i) \oplus (X'_i, p'_i)$ is isomorphic to an object Y_i in \mathcal{C} , and such that the sequence

$$Y_0 \rightarrow Y_1 \rightarrow Y_2$$

is exact in \mathcal{C} . The category \mathcal{C}^{ic} is an idempotent complete exact category. The embedding $\mathcal{C} \hookrightarrow \mathcal{C}^{ic}$ is fully exact. This embedding is 2-universal in the category of exact functors $\mathcal{C} \rightarrow \mathcal{D}$ with \mathcal{D} idempotent complete.

2.2.2. Exact, Full Sub-Categories. We recall here three conditions which may hold for exact, full sub-categories $\mathcal{C} \subset \mathcal{D}$. As explained to us by T. Bühler, if a sub-category satisfies all three of these conditions, then it is “left s-filtering” in the sense of Schlichting [27, Definition 1.5]. Left s-filtering sub-categories play many of the same roles for exact categories as Serre sub-categories play for abelian categories.

Definition 2.10. A full sub-category \mathcal{C} of an exact category \mathcal{D} is *closed under extensions* if, for every exact sequence

$$X \hookrightarrow F \twoheadrightarrow Z$$

with X and Z in \mathcal{C} , we have $F \in \mathcal{C}$ as well.

²This example relies on the difference, for disconnected spaces, between a module being free and it being free on each component.

The following lemma is a simple exercise in the definitions.

Lemma 2.11. *Let \mathcal{D} be an exact category. Let $\mathcal{C} \subset \mathcal{D}$ be closed under extensions. Define a sequence in \mathcal{C} to be exact if it is exact in \mathcal{D} . This endows \mathcal{C} with the structure of an exact category.*

Definition 2.12. An exact, full sub-category $\mathcal{C} \subset \mathcal{D}$ is *left special* if, for every admissible epic $F \twoheadrightarrow X$ in \mathcal{D} with $X \in \mathcal{C}$, there exists a commutative diagram in \mathcal{D}

$$\begin{array}{ccccc} Z & \hookrightarrow & Y & \twoheadrightarrow & X \\ \downarrow & & \downarrow & & \parallel \\ G & \hookrightarrow & F & \twoheadrightarrow & X \end{array}$$

in which the top row is an exact sequence in \mathcal{C} and the bottom row is an exact sequence in \mathcal{D} . We say \mathcal{C} is *right special* if \mathcal{C}^{op} is *left special* in \mathcal{D}^{op} .

Remark 2.13. We adapt the terminology “left special” from Schlichting [27, Definition 1.5]. Right special sub-categories were previously studied by Keller as sub-categories satisfying “Condition C2” [15, Section 12.1].

Lemma 2.14. *Left special sub-categories are closed under extensions.*

Proof. Let $\mathcal{C} \subset \mathcal{D}$ be a left special sub-category. Let

$$X \hookrightarrow F \twoheadrightarrow Z$$

be an exact sequence in \mathcal{D} with X and Z in \mathcal{C} . By assumption, there exists a commuting diagram in \mathcal{D}

$$\begin{array}{ccccc} V & \hookrightarrow & Y & \twoheadrightarrow & Z \\ \downarrow & & \downarrow & & \parallel \\ X & \hookrightarrow & F & \twoheadrightarrow & Z \end{array}$$

in which the top row is an exact sequence in \mathcal{C} and the bottom row is an exact sequence in \mathcal{D} . Pushing out the admissible monic $V \hookrightarrow Y$ along the map $V \twoheadrightarrow X$, we obtain a second commuting diagram in \mathcal{D}

$$\begin{array}{ccccc} X & \hookrightarrow & Y' & \twoheadrightarrow & Z \\ \parallel & & \downarrow & & \parallel \\ X & \hookrightarrow & F & \twoheadrightarrow & Z \end{array}$$

in which the top row is an exact sequence in \mathcal{C} and the bottom row is an exact sequence in \mathcal{D} . The 5-Lemma [9, Corollary 3.2] implies that the map $Y' \twoheadrightarrow F$ is an isomorphism. \square

Definition 2.15. An exact, full sub-category $\mathcal{C} \subset \mathcal{D}$ is *left filtering* if every morphism $X \twoheadrightarrow F$ in \mathcal{D} , with $X \in \mathcal{C}$, factors through an admissible monic $X' \hookrightarrow F$

with $X' \in \mathcal{C}$:

$$\begin{array}{ccc} X & \xrightarrow{\quad} & F \\ & \searrow & \uparrow \\ & & X' \end{array}$$

We say \mathcal{C} is *right filtering* if \mathcal{C}^{op} is left filtering in \mathcal{D}^{op} .

Definition 2.16 (Schlichting [27]). An exact, full sub-category $\mathcal{C} \subset \mathcal{D}$ is *left special filtering*, or “left s-filtering” for short, if it is left special and left filtering. We say \mathcal{C} is *right s-filtering* if \mathcal{C}^{op} is left s-filtering in \mathcal{D}^{op} .

Remark 2.17. We differ slightly from Schlichting in our presentation of left s-filtering. See Appendix A for a proof that the definitions agree.

We record the following elementary observations.

Lemma 2.18. *Let $\mathcal{C} \subset \mathcal{D} \subset \mathcal{D}'$ be a chain of exact, fully faithful embeddings.*

- (1) *If \mathcal{C} is closed under extensions in \mathcal{D}' , then \mathcal{C} is closed under extensions in \mathcal{D} .*
- (2) *If \mathcal{C} is left special in \mathcal{D}' , then \mathcal{C} is left special in \mathcal{D} .*
- (3) *If \mathcal{C} is left special in \mathcal{D}' and left filtering in \mathcal{D} , then \mathcal{C} is left s-filtering in \mathcal{D} .*

The conditions of left filtering and left s-filtering play a role in forming quotient categories. We summarize the key facts, which we learned from Schlichting and Bühler, here.

Proposition 2.19. *Let $\mathcal{C} \subset \mathcal{D}$ be a full sub-category of an exact category. Denote by Σ_e the collection of admissible epics in \mathcal{D} with kernel in \mathcal{C} .*

- (1) *Then Σ_e admits a calculus of left fractions in \mathcal{D} if and only if \mathcal{C} is left filtering and closed under extensions in \mathcal{D} .*
- (2) *Denote by $\mathcal{D}[\Sigma_e^{-1}]$ the localization of \mathcal{D} at Σ_e . If \mathcal{C} is left s-filtering in \mathcal{D} , then every admissible monic in \mathcal{D} with cokernel in \mathcal{C} is invertible in $\mathcal{D}[\Sigma_e^{-1}]$. In this case, we alternately denote $\mathcal{D}[\Sigma_e^{-1}]$ by \mathcal{D}/\mathcal{C} .*
- (3) *If $\mathcal{C} \subset \mathcal{D}$ is left s-filtering, then \mathcal{D}/\mathcal{C} has a natural structure of an exact category in which a sequence is exact if and only if it is the image of an exact sequence under the map $\mathcal{D} \rightarrow \mathcal{D}/\mathcal{C}$.*

The first two statements are due to Bühler. Once one knows the statements, the proofs are straightforward; we leave them to the interested reader. The third statement is due to Schlichting; we refer the reader to [27, Proposition 1.16] for the proof.

2.3. Left Exact Presheaves.

Definition 2.20. Let \mathcal{C} be an exact category. Denote by $\text{Lex}(\mathcal{C})$ the category of left-exact presheaves of abelian groups, i.e. functors $F: \mathcal{C}^{op} \rightarrow \text{Ab}$ such that if

$$X \hookrightarrow Y \twoheadrightarrow Z$$

is a short exact sequence in \mathcal{C} , then the sequence of abelian groups

$$0 \longrightarrow F(Z) \longrightarrow F(Y) \longrightarrow F(X)$$

is exact.

The category $\text{Lex}(\mathcal{C})$ is familiar in many contexts.

Lemma 2.21. *Let R be a ring. Denote by $\text{Mod}(R)$ the category of (left) R -modules. The assignment*

$$M \mapsto \text{hom}_R(-, M)$$

defines an equivalence of categories

$$\text{Mod}(R) \xrightarrow{\simeq} \text{Lex}(P^f(R))$$

Proof. The inverse equivalence can be described as follows. Let F be a left-exact presheaf of abelian groups. Denote by $P^f(R) \downarrow F$ the category whose objects are morphisms of left exact presheaves $N \rightarrow F$ where N is a finitely generated projective left R -module. Morphisms of $P^f(R) \downarrow F$ are commuting triangles over F . The inverse equivalence $\text{Lex}(P^f(R)) \rightarrow \text{Mod}(R)$ is given on objects by

$$F \mapsto \text{colim}_{P^f(R) \downarrow F} N$$

where the colimit is formed in the category of R -modules. Because finitely generated projective modules are in fact finitely presented, we can conclude that the above functor is a left and a right inverse. \square

Similarly, we have the following.

Lemma 2.22. *Let X be a Noetherian scheme, and denote by $\text{Coh}(X)$ and $\text{QCoh}(X)$ the categories of coherent and quasi-coherent sheaves on X . Then $\text{QCoh}(X) \simeq \text{Lex}(\text{Coh}(X))$.*

These examples anticipate two other characterizations of $\text{Lex}(\mathcal{C})$:

- (1) Keller [14, Appendix A], following Freyd and Quillen, exhibits $\text{Lex}(\mathcal{C})$ as a localization, with respect to a Serre sub-category, of the category $\text{Ab}^{\mathcal{C}^{op}}$ of presheaves of abelian groups on \mathcal{C} . Define a presheaf F of abelian groups to be *effaceable* if, for every $Y \in \mathcal{C}$ and every section

$$Y \longrightarrow F$$

there exists an admissible epic in \mathcal{C} , as in the figure below, whose target is Y and such that the restriction of the section along this epic is 0.

$$\begin{array}{ccc} X & & \\ \downarrow & \searrow 0 & \\ Y & \longrightarrow & F \end{array}$$

Effaceable presheaves form a Serre sub-category of $\text{Ab}^{\mathcal{C}^{op}}$, and $\text{Lex}(\mathcal{C})$ is the associated Serre quotient. This exhibits $\text{Lex}(\mathcal{C})$ as an abelian category.

- (2) Thomason and Trobaugh [30, Appendix A], following Laumon and Gabriel, observe that the admissible epics in \mathcal{C} define a pre-topology.³ They exhibit $\text{Lex}(\mathcal{C})$ as the category of sheaves of abelian groups with respect to this topology. This shows that the inclusion of $\text{Lex}(\mathcal{C})$ into $\text{Ab}^{\mathcal{C}^{op}}$ preserves and reflects limits, and that the localization

$$\text{Ab}^{\mathcal{C}^{op}} \longrightarrow \text{Lex}(\mathcal{C})$$

³See also [9, Appendix A].

preserves finite limits.

Proposition 2.23. *Every exact category \mathcal{C} is left special in $\mathbf{Lex}(\mathcal{C})$.*

Proof. Let $F \twoheadrightarrow Z$ be an epic in $\mathbf{Lex}(\mathcal{C})$ with $Z \in \mathcal{C}$. The cokernel, in $\mathbf{Ab}^{\mathcal{C}^{op}}$, of this map is an effaceable presheaf,⁴ so there exists an admissible epic $Y \twoheadrightarrow Z$ in \mathcal{C} fitting into a commuting triangle

$$\begin{array}{ccc} & Y & \\ \swarrow & \downarrow & \\ F & \twoheadrightarrow & Z \end{array}.$$

Taking the kernels of the maps to Z , we obtain the desired commuting diagram

$$\begin{array}{ccccc} X & \hookrightarrow & Y & \twoheadrightarrow & Z \\ \downarrow & & \downarrow & & \parallel \\ G & \hookrightarrow & F & \twoheadrightarrow & Z \end{array}$$

□

Remark 2.24. In general, \mathcal{C} is very far from being left filtering in $\mathbf{Lex}(\mathcal{C})$. As a basic example, consider the category $F^f(\mathbb{Z})$ of finitely generated free abelian groups. The category $\mathbf{Lex}(F^f(\mathbb{Z}))$ is equivalent to the category of abelian groups. The map $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2$ does not factor through a monic from a free abelian group.

The proof of Lemma 2.21 contains a basic construction we will use again.

Definition 2.25. Let \mathcal{C} be an exact category, and let F be a left exact presheaf. Define the *category of elements* $\mathcal{C} \downarrow F$ of F as follows.

- (1) Objects are maps $X \rightarrow F$ in $\mathbf{Lex}(\mathcal{C})$ from an object $X \in \mathcal{C}$ to F .
- (2) A morphism from $X \rightarrow F$ to $Y \rightarrow F$ is a commuting triangle

$$\begin{array}{ccc} X & \xrightarrow{\quad} & Y \\ & \searrow & \swarrow \\ & F & \end{array}$$

The following is a standard fact about categories of elements (e.g. see [6, Theorem 2.15.6]).

Lemma 2.26. *Let F be a left exact presheaf on \mathcal{C} . The canonical map*

$$\operatorname{colim}_{\mathcal{C} \downarrow F} X \longrightarrow F$$

is an isomorphism, where the colimit is taken in $\mathbf{Lex}(\mathcal{C})$.

⁴If we wished to use Thomason's description of $\mathbf{Lex}(\mathcal{C})$, the existence of this triangle follows from the observation that the map $F \twoheadrightarrow Z$ is an epimorphism of sheaves. Therefore there exists a cover $Z' \twoheadrightarrow Z$, i.e. an admissible epic, which factors through the map from F .

3. ADMISSIBLE IND-OBJECTS

In this section, we develop the properties of admissible Ind-objects in an exact category \mathcal{C} . Admissible Ind-objects sit in relation to objects of \mathcal{C} as the R -module $R[t]$ sits in relation to finitely generated free R -modules.⁵ More generally, the results of this section should be viewed as an elaboration, in the setting of exact categories, of the dual to Artin–Mazur [2, Appendix 2].

3.1. The Category of Admissible Ind-Objects.

Definition 3.1. Let I and J be directed posets. A functor $\varphi : I \rightarrow J$ is *final* if for every $j \in J$, there exists $i \in I$ with $j \leq \varphi(i)$.

Definition 3.2. Let \mathcal{C} be an exact category. Let κ be an infinite cardinal. An *admissible Ind-diagram of size at most κ* is a functor

$$I \xrightarrow{X} \mathcal{C}$$

such that I is a directed poset of cardinality at most κ , and such that X takes any arrow in I to an admissible monic in \mathcal{C} . Morphisms of admissible Ind-diagrams are 2-commuting triangles

$$\begin{array}{ccc} I & \xrightarrow{\varphi} & J \\ & \searrow X \quad \alpha \quad \swarrow Y & \\ & \mathcal{C} & \end{array} .$$

Denote the category of admissible Ind-diagrams of cardinality at most κ by $\text{Dir}_\kappa^a(\mathcal{C})$.

An admissible Ind-diagram defines a left-exact presheaf by the assignment

$$Y \longmapsto \text{colim}_I \text{hom}_{\mathcal{C}}(Y, X_i) .$$

This extends to a functor

$$(3.1) \quad \text{Dir}_\kappa^a(\mathcal{C}) \xrightarrow{\widehat{(-)}} \text{Lex}(\mathcal{C}) .$$

Definition 3.3. Define the category $\text{Ind}_\kappa^a(\mathcal{C})$ of *admissible Ind-objects in \mathcal{C} of size at most κ* to be the full sub-category of $\text{Lex}(\mathcal{C})$ consisting of objects in the essential image of 3.1.

We can also omit the cardinality bound κ .

Definition 3.4. Denote by $\text{Dir}^a(\mathcal{C})$ the (large) category of admissible Ind-diagrams of arbitrary cardinality. Denote by $\text{Ind}^a(\mathcal{C})$ the full sub-category of $\text{Lex}(\mathcal{C})$ consisting of objects in the essential image of $\text{Dir}^a(\mathcal{C})$.

Remark 3.5. Countable admissible Ind-objects have been studied for some time, for instance in [14, Appendix A], or [27]. In Section 3.2.5, we show that for $\kappa = \aleph_0$, the category $\text{Ind}_{\aleph_0}^a(\mathcal{C})$ recovers Keller’s treatment (Proposition 3.17). By allowing for uncountable admissible Ind-objects, we can treat examples such as the category of projective modules over a ring in terms of admissible Ind-objects (see Section 3.3).

⁵For definiteness, the R -module $R[t]$ is the direct sum $\bigoplus_{n \in \mathbb{N}} R$.

If $X: I \longrightarrow \mathcal{C}$ and $Y: J \longrightarrow \mathcal{C}$ are admissible Ind-diagrams in \mathcal{C} , then the definition ensures that

$$\mathrm{hom}_{\mathrm{Ind}_\kappa^a(\mathcal{C})}(\widehat{X}, \widehat{Y}) \cong \lim_I \mathrm{colim}_J \mathrm{hom}_{\mathcal{C}}(X_i, Y_j)$$

In particular, we see that, for any map $f: X \longrightarrow \widehat{Y}$ in $\mathrm{Ind}_\kappa^a(\mathcal{C})$ with $X \in \mathcal{C}$, and for any admissible diagram $Y: J \longrightarrow \mathcal{C}$ representing an \widehat{Y} , there exists $j \in J$ such that f factors through some $Y_j \hookrightarrow \widehat{Y}$. This is a key property in what follows, and part of the general phenomenon that objects of \mathcal{C} are *finitely presentable* in $\mathrm{Lex}(\mathcal{C})$.

Remark 3.6. We could define an admissible Ind-diagram to be a map $X: I \rightarrow \mathcal{C}$ such that I is a filtering category, rather than a directed poset. However, this would give an equivalent notion of admissible Ind-object. Indeed, the image of such a diagram X is a directed poset because the maps $X_i \rightarrow X_j$ are all monic.

Theorem 3.7. *For any infinite cardinal κ , the category $\mathrm{Ind}_\kappa^a(\mathcal{C})$ is closed under extensions in $\mathrm{Lex}(\mathcal{C})$. In particular, $\mathrm{Ind}_\kappa^a(\mathcal{C})$ admits a canonical structure of an exact category. Further, if \mathcal{C} is weakly idempotent complete, then $\mathrm{Ind}_\kappa^a(\mathcal{C})$ is left special in $\mathrm{Lex}(\mathcal{C})$. The same is true for $\mathrm{Ind}^a(\mathcal{C})$.*

The following lemma contains the core of the proof of the theorem.

Lemma 3.8. *Let $F \twoheadrightarrow \widehat{Z}$ be an epic in $\mathrm{Lex}(\mathcal{C})$ with $\widehat{Z} \in \mathrm{Ind}_\kappa^a(\mathcal{C})$. For any $Z: I \longrightarrow \mathcal{C}$ in $\mathrm{Dir}_\kappa^a(\mathcal{C})$ representing \widehat{Z} , there exists a morphism in $\mathrm{Dir}_\kappa^a(\mathcal{C})$*

$$\begin{array}{ccc} K & \xrightarrow{\varphi} & J \\ & \searrow \alpha \quad \swarrow Z & \\ & \mathcal{C} & \end{array} \quad \begin{array}{c} Y \\ \nearrow \end{array}$$

such that:

- (1) the map φ is final,
- (2) for all $k \in K$, the map $\alpha_k: Y_k \longrightarrow Z_{\varphi(k)}$ is an admissible epic, and
- (3) the induced map $\widehat{Y} \twoheadrightarrow \widehat{Z}$ factors through the map $F \twoheadrightarrow \widehat{Z}$ as in

$$\begin{array}{ccc} & \widehat{Y} & \\ & \downarrow & \\ F & \twoheadrightarrow & \widehat{Z} \end{array}$$

We also need the following minor restatement of the duals of [2, Proposition A.3.1] and [2, Corollary A.3.2].⁶

Lemma 3.9 (Straightening Morphisms). *Let $X: I \longrightarrow \mathcal{C}$ and $Y: J \longrightarrow \mathcal{C}$ be diagrams in \mathcal{C} where I and J are directed posets of cardinality at most κ , and where, for all $i \leq i'$ in I and $j \leq j'$ in J , the maps $X_i \longrightarrow X_{i'}$ and $Y_j \longrightarrow Y_{j'}$ are (not necessarily admissible) monics. Denote by \widehat{X} and \widehat{Y} the colimits of these diagrams in $\mathrm{Lex}(\mathcal{C})$. Given a map $f: \widehat{X} \rightarrow \widehat{Y}$, define $I \downarrow_{\mathcal{C}} J$ to be the category whose objects are triples*

⁶The only difference, besides notation, is that our assumption that all maps in the diagrams are monic allows us to replace “ U -small” by a definite cardinality bound on the size of $I \downarrow_{\mathcal{C}} J$.

(i, j, α_{ij}) where $i \in I$, $j \in J$ and α_{ij} is a morphism which fits into a commuting square (in $\text{Lex}(\mathcal{C})$)

$$(3.2) \quad \begin{array}{ccc} X_i & \xrightarrow{\alpha_{ij}} & Y_j \\ \downarrow & & \downarrow \\ \widehat{X} & \xrightarrow{f} & \widehat{Y} \end{array}$$

Then $I \downarrow_{\mathcal{C}} J$ is a directed poset of cardinality at most κ .

Further, define $\varphi: I \downarrow_{\mathcal{C}} J \rightarrow I$ and $\psi: I \downarrow_{\mathcal{C}} J \rightarrow J$ to be the projections in the I and J factors. Both φ and ψ are final, and they give rise to a span in $\text{Dir}_{\kappa}^a(\mathcal{C})$

$$\begin{array}{ccccc} I & \xleftarrow{\varphi} & I \downarrow_{\mathcal{C}} J & \xrightarrow{\psi} & J \\ & \searrow X & \downarrow \Rightarrow^{\alpha} & \nearrow Y & \\ & & \mathcal{C} & & \end{array}$$

in which the triangle on the left strictly commutes, and the component of α at (i, j, α_{ij}) is given by α_{ij} .

In particular, any map of admissible Ind-objects can be straightened to a map of admissible Ind-diagrams.

Proof of Theorem 3.7. We use the lemmas to show that $\text{Ind}_{\kappa}^a(\mathcal{C}) \subset \text{Lex}(\mathcal{C})$ is closed under extensions. If \mathcal{C} is weakly idempotent complete, our argument will also show that $\text{Ind}_{\kappa}^a(\mathcal{C})$ is left special in $\text{Lex}(\mathcal{C})$.

Let

$$\widehat{X} \hookrightarrow F \twoheadrightarrow \widehat{Z}$$

be a short exact sequence in $\text{Lex}(\mathcal{C})$ where \widehat{X} and \widehat{Z} are in $\text{Ind}_{\kappa}^a(\mathcal{C})$. Let $X: I \rightarrow \mathcal{C}$ and $Z: J \rightarrow \mathcal{C}$ be admissible Ind-diagrams of cardinality at most κ representing \widehat{X} and \widehat{Z} . Lemma 3.8 guarantees the existence of a map of admissible Ind-diagrams of cardinality at most κ

$$\begin{array}{ccc} K & \xrightarrow{\varphi} & J \\ & \searrow Y' \quad \Rightarrow^{\alpha} \quad \nearrow Z & \\ & & \mathcal{C} \end{array}$$

such that all of the components of α are admissible epics and such that the induced map $\widehat{Y'} \twoheadrightarrow \widehat{Z}$ factors through the map $F \twoheadrightarrow \widehat{Z}$. Define a directed diagram

$$\begin{array}{ccc} K & \xrightarrow{X'} & \mathcal{C} \\ k \longmapsto & \ker(\alpha_k) & \end{array}$$

The maps in this diagram are the monics induced by the universal property of kernels.⁷ This diagram fits into an exact sequence with the admissible Ind-diagrams

⁷Note that, while the maps in this diagram are monics, we do not claim that they are admissible monics; in general, this is only the case under the additional assumption that \mathcal{C} is weakly idempotent complete.

Y' and Z

$$\begin{array}{ccccc}
 K & \xlongequal{\quad} & K & \xrightarrow{\quad} & J \\
 & \searrow & \downarrow & \swarrow & \\
 & & Y' & & \\
 & \swarrow & \downarrow & \searrow & \\
 & & \mathcal{C} & &
 \end{array}
 \begin{array}{c}
 \beta \\
 \Rightarrow \\
 X'
 \end{array}
 \begin{array}{c}
 \alpha \\
 \Rightarrow \\
 Z
 \end{array}$$

where the map $K \rightarrow J$ is final, the components of α are admissible epics, and the components of β are admissible monics. Taking the colimit of these diagrams in $\mathbf{Lex}(\mathcal{C})$, we obtain a commuting diagram with exact rows

$$\begin{array}{ccccc}
 \widehat{X}' & \hookrightarrow & \widehat{Y}' & \twoheadrightarrow & \widehat{Z} \\
 \downarrow & & \downarrow & & \parallel \\
 \widehat{X} & \hookrightarrow & F & \twoheadrightarrow & \widehat{Z}
 \end{array}$$

Note that the map from \widehat{X}' to \widehat{X} is induced by the universal property of kernels. Using Lemma 3.9, we straighten this map to a span

$$\begin{array}{ccccc}
 K & \longleftarrow & K \downarrow_{\mathcal{C}} I & \longrightarrow & I \\
 & \searrow & \downarrow & \swarrow & \\
 & & \mathcal{C} & &
 \end{array}
 \begin{array}{c}
 X' \\
 \gamma \\
 X
 \end{array}$$

and define

$$\begin{array}{c}
 K \downarrow_{\mathcal{C}} I \xrightarrow{Y} \mathcal{C} \\
 (k, i, \gamma_{ki}) \longmapsto X_i \cup_{X'_k} Y'_k
 \end{array}$$

A map $(k, i, \gamma_{ki}) \rightarrow (\ell, j, \eta_{\ell j})$ induces a map $Y_{(k, i, \gamma_{ki})} \rightarrow Y_{(\ell, j, \eta_{\ell j})}$ by the universal property of pushouts.

The diagram Y is admissible. Indeed, for each arrow $(k, i, \gamma_{ki}) \rightarrow (\ell, j, \eta_{\ell j})$ in $K \downarrow_{\mathcal{C}} I$, we have a diagram with exact rows

$$\begin{array}{ccccc}
 X_i & \hookrightarrow & Y_{(k, i, \gamma_{ki})} & \twoheadrightarrow & Z_{\varphi(k)} \\
 \downarrow & & \downarrow & & \downarrow \\
 X_j & \hookrightarrow & Y_{(\ell, j, \eta_{\ell j})} & \twoheadrightarrow & Z_{\varphi(\ell)}
 \end{array}$$

The left and right vertical arrows are admissible monics. Therefore the middle vertical arrow is an admissible monic as well [9, Corollary 3.2].

Passing back to the associated admissible Ind-objects, we obtain a commuting diagram in $\mathbf{Lex}(\mathcal{C})$ with exact rows

$$\begin{array}{ccccc}
 \widehat{X} & \hookrightarrow & \widehat{Y} & \twoheadrightarrow & \widehat{Z} \\
 \parallel & & \downarrow & & \parallel \\
 \widehat{X} & \hookrightarrow & F & \twoheadrightarrow & \widehat{Z}
 \end{array}$$

The 5-Lemma shows that the map from \widehat{Y} to F is an isomorphism. We conclude that $\text{Ind}_\kappa^a(\mathcal{C})$ is closed under extensions in $\text{Lex}(\mathcal{C})$, and is therefore a fully exact sub-category.

If \mathcal{C} is weakly idempotent complete, let

$$G \hookrightarrow F \twoheadrightarrow \widehat{Z}$$

be a short exact sequence in $\text{Lex}(\mathcal{C})$ with $\widehat{Z} \in \text{Ind}_\kappa^a(\mathcal{C})$. The construction above yields a commuting diagram with exact rows

$$\begin{array}{ccccc} \widehat{X}' & \hookrightarrow & \widehat{Y}' & \twoheadrightarrow & \widehat{Z} \\ \downarrow & & \downarrow & & \parallel \\ G & \hookrightarrow & F & \twoheadrightarrow & \widehat{Z} \end{array}$$

Because \mathcal{C} is weakly idempotent complete, the maps in the diagram $X': K \longrightarrow \mathcal{C}$ are all admissible monics. We conclude that $\widehat{X}' \in \text{Ind}_\kappa^a(\mathcal{C})$, and that $\text{Ind}_\kappa^a(\mathcal{C})$ is left special in $\text{Lex}(\mathcal{C})$. \square

Proof of Lemma 3.8. Our argument follows Keller [14, Appendix B]. The only change is that, in considering $\kappa > \aleph_0$, we need to work with directed posets rather than only linear orders. Let

$$F \twoheadrightarrow \widehat{Z}$$

be an epic in $\text{Lex}(\mathcal{C})$ with $\widehat{Z} \in \text{Ind}_\kappa^a(\mathcal{C})$. Let $Z: J \longrightarrow \mathcal{C}$ be an admissible Ind-diagram representing \widehat{Z} . For all $j \in J$, we can pull back F along the map $Z_j \hookrightarrow \widehat{Z}$. Denote the resulting epic by $F_j \twoheadrightarrow Z_j$. The category \mathcal{C} is left special in $\text{Lex}(\mathcal{C})$, so there exists an admissible epic $Y_j \twoheadrightarrow Z_j$ in \mathcal{C} such that we have a triangle

$$\begin{array}{ccc} & Y_j & \\ \swarrow & \downarrow & \\ F_j & \twoheadrightarrow & Z_j \end{array}$$

We use these triangles to construct an admissible Ind-diagram

$$K \xrightarrow{Y} \mathcal{C}$$

with the desired properties.

Let K be the category whose objects are finite directed sub-posets $J_a \subset J$. Morphisms in K are inclusions of sub-diagrams. We first observe that $|K| \leq \coprod_{n \in \mathbb{N}} |J|^n \leq \kappa$.

We now show that K is directed. Because J is directed, for any finite collection $\{j_i\}_{i=0}^n$ of objects of J , there exists $j \in J$ such that $j_i \leq j$ for $i = 0, \dots, n$. Given any two objects $J_{a_0}, J_{a_1} \in K$, their union in J consists of a finite collection of objects of J . Taking an upper bound on this diagram, we obtain a finite directed sub-poset of J which contains both J_{a_0} and J_{a_1} .

The assignment

$$J_a \longmapsto \max J_a$$

defines a final map from K to J .

Define an admissible Ind-diagram by

$$\begin{aligned} K &\xrightarrow{Y} \mathcal{C} \\ J_a &\longmapsto \bigoplus_{j \in J_a} Y_j \end{aligned}$$

Morphisms are given by inclusions of summands.

Define a natural transformation

$$\begin{array}{ccc} K & \xrightarrow{\quad} & J \\ & \searrow Y \quad \alpha \quad \swarrow Z & \\ & \mathcal{C} & \end{array}$$

by defining the restriction of the component α_{J_a} to the Y_j -summand of Y_{J_a} to be the composite

$$Y_j \longrightarrow Z_j \longrightarrow \cdots \longrightarrow Z_{\max J_a}.$$

The α_{J_a} are admissible epics for all $J_a \in K$. Indeed, consider the shear map

$$\bigoplus_{j \in J_a} Y_j \xrightarrow{\sigma} \bigoplus_{j \in J_a} Y_j.$$

i.e. the map whose restriction to the Y_j summand is given by

$$\sum_{j' \geq j} (Y_j \longrightarrow Y_{j'})$$

The shear map is an isomorphism, because $|J_a|$ is finite. It factors α_{J_a} as

$$\bigoplus_{j \in J_a} Y_j \longrightarrow \bigoplus_{j \in J_a} Z_j \xrightarrow{\sigma} \bigoplus_{j \in J_a} Z_j \longrightarrow Z_{\max J_a}$$

where the last map is the projection onto the factor $Z_{\max J_a}$. Each map in this factorization is an admissible epic in \mathcal{C} , so α_{J_a} is an admissible epic as well.

We now define a co-cone on the diagram Y with co-cone point F . For $J_a \in K$, the restriction of the map $Y_{J_a} \longrightarrow F$ to the Y_j summand is given by the composite

$$Y_j \longrightarrow F_j \longrightarrow \cdots \longrightarrow F_{\max J_a} \longrightarrow F$$

By the universal property of colimits, this determines a unique map $\hat{Y} \longrightarrow F$ fitting into the desired commuting triangle. \square

3.2. Properties of Admissible Ind-Objects.

3.2.1. \mathcal{C} as an Exact, Full Sub-Category of $\text{Ind}_{\kappa}^a(\mathcal{C})$.

Proposition 3.10. *The category \mathcal{C} is left s -filtering in $\text{Ind}_{\kappa}^a(\mathcal{C})$.*

In Proposition 2.23 we showed that \mathcal{C} is left special in $\text{Lex}(\mathcal{C})$. To prove the proposition, it suffices to show that \mathcal{C} is left filtering in $\text{Ind}_{\kappa}^a(\mathcal{C})$. This is a consequence of the following.

Lemma 3.11. *Let $X: I \longrightarrow \mathcal{C}$ be an admissible Ind-diagram. Then for any $i \in I$, the map $X_i \longrightarrow \hat{X}$ is an admissible monic in $\text{Ind}_{\kappa}^a(\mathcal{C})$.*

Proof. Denote by $I_i \subset I$ the sub-poset consisting of all $j \in I$ such that $i \leq j$. The inclusion $I_i \hookrightarrow I$ is final. The diagram

$$\begin{aligned} I_i &\xrightarrow{X/X_i} \mathcal{C} \\ j &\longmapsto X_j/X_i \end{aligned}$$

is admissible, because admissible monics push out. It fits into a sequence of admissible Ind-diagrams

$$\begin{array}{ccccc}
 I_i & \xlongequal{\quad} & I_i & \xlongequal{\quad} & I_i \\
 & \searrow & \downarrow \alpha & \nearrow \beta & \\
 & X_i & \xRightarrow{\quad} X & \xRightarrow{\quad} X/X_i & \\
 & & \downarrow & & \\
 & & \mathcal{C} & &
 \end{array}$$

This sequence defines an exact sequence of admissible Ind-objects, whose first map is the canonical map $X_i \longrightarrow \widehat{Y}$. \square

Proof of Proposition 3.10. Let $f: X \longrightarrow \widehat{Y}$ be a morphism in $\text{Ind}_\kappa^a(\mathcal{C})$ with $X \in \mathcal{C}$. For any diagram $Y: J \longrightarrow \mathcal{C}$ presenting \widehat{Y} , there exists $j \in J$ such that f factors through the map $Y_j \hookrightarrow \widehat{Y}$. This map is an admissible monic by the previous lemma. We conclude that \mathcal{C} is left filtering, and thus left s-filtering, in $\text{Ind}_\kappa^a(\mathcal{C})$. \square

3.2.2. Exact Sequences of Admissible Ind-Objects.

Proposition 3.12. *The categories $\text{Ind}_\kappa^a(\mathcal{EC})$ and $\mathcal{E}\text{Ind}_\kappa^a(\mathcal{C})$ are canonically equivalent.*

Proof. The category \mathcal{EC} has a canonical exact structure [12] in which admissible monics (epics) are maps of exact sequences which are admissible monics (epics) at each term in the sequence.

An admissible Ind-diagram in \mathcal{EC} consists of a pair of maps of admissible Ind-diagrams in \mathcal{C}

$$\begin{array}{ccccc}
 I & \xlongequal{\quad} & I & \xlongequal{\quad} & I \\
 & \searrow & \downarrow \alpha & \nearrow \beta & \\
 & X & \xRightarrow{\quad} Y & \xRightarrow{\quad} Z & \\
 & & \downarrow & & \\
 & & \mathcal{C} & &
 \end{array}$$

where the components of α are admissible monics, and those of β are admissible epics.

Because directed colimits in $\text{Lex}(\mathcal{C})$ preserve kernels and cokernels, an admissible Ind-diagram of exact sequences in \mathcal{C} canonically defines an exact sequence of admissible Ind-objects. This extends to a faithful functor

$$(3.3) \quad \text{Ind}_\kappa^a(\mathcal{EC}) \longrightarrow \mathcal{E}\text{Ind}_\kappa^a(\mathcal{C})$$

The construction in the proof of Theorem 3.7 implies that this functor is essentially surjective: every short exact sequence of admissible Ind-objects arises as the colimit of a directed diagram of short exact sequences in \mathcal{C} .

It remains to show that the functor 3.3 is full. Let

$$(3.4) \quad \begin{array}{ccccc}
 \widehat{X}_0 & \hookrightarrow & \widehat{Y}_0 & \twoheadrightarrow & \widehat{Z}_0 \\
 \downarrow g_X & & \downarrow g_Y & & \downarrow g_Z \\
 \widehat{X}_1 & \hookrightarrow & \widehat{Y}_1 & \twoheadrightarrow & \widehat{Z}_1
 \end{array}$$

be a morphism in $\mathcal{E}\text{Ind}_\kappa^a(\mathcal{C})$. Let

$$\begin{array}{ccccc} I & \xlongequal{\quad} & I & \xlongequal{\quad} & I \\ & \searrow & \downarrow \alpha_1 & \swarrow \beta_1 & \\ & X_1 & Y_1 & Z_1 & \\ & & \downarrow & & \\ & & \mathcal{C} & & \end{array}$$

be an admissible Ind-diagram of exact sequences representing the bottom row of 3.4. Straightening (Lemma 3.9) allows us to represent the map g_Z as a 2-commuting triangle

$$\begin{array}{ccc} J & \xrightarrow{\quad} & I \\ & \searrow \gamma_Z \swarrow & \\ & Z_0 & Z_1 \\ & & \downarrow \\ & & \mathcal{C} \end{array}$$

We can represent the admissible epic $\widehat{Y}_0 \rightarrow \widehat{Z}_0$ as a 2-commuting triangle

$$\begin{array}{ccc} K & \xrightarrow{\quad} & J \\ & \searrow \beta'_0 \swarrow & \\ & Y'_0 & Z_0 \\ & & \downarrow \\ & & \mathcal{C} \end{array}$$

where the map $K \rightarrow J$ is final.

We straighten g_Y to a 2-commuting triangle

$$\begin{array}{ccc} L & \xrightarrow{\varphi} & I \\ & \searrow \gamma_{Y'} \swarrow & \\ & Y'_0 & Y_1 \\ & & \downarrow \\ & & \mathcal{C} \end{array}$$

The straightening construction also produces a final map $\psi': L \rightarrow K$. Denote by ψ the composite $L \xrightarrow{\psi'} K \rightarrow J$. The 2-commuting triangle above embeds in a 2-commuting pyramid

$$\begin{array}{ccccc} L & \xlongequal{\quad} & L & & L \\ & \searrow Y'_0 & \downarrow \beta'_0 \psi' & \swarrow Z_0 \psi & \\ & & \mathcal{C} & & \\ & \swarrow \gamma_{Y'} & \downarrow \gamma_Z \psi & \searrow & \\ & Y_1 & \downarrow \beta_1 & Z_1 & \\ I & \xlongequal{\quad} & I & & I \end{array} \quad .$$

Indeed, for every $l \in L$, we have a pair of commuting squares of admissible Ind-objects

$$\begin{array}{ccc} Y'_{0,l} & \begin{array}{c} \xrightarrow{\gamma_{Z,\psi(l)} \circ \beta'_{0,\psi'(l)}} \\ \xrightarrow{\beta_{1,\varphi(l)} \circ \gamma_{Y',l}} \end{array} & Z_{1,\varphi(l)} \\ \downarrow & & \downarrow \\ \widehat{Y}_0 & \longrightarrow & \widehat{Z}_1 \end{array}$$

Because the map $Z_{1,\varphi(l)} \longrightarrow \widehat{Z}_1$ is monic, we see that $\gamma_{Z,\psi(l)} \circ \beta'_{0,\psi'(l)} = \beta_{1,\varphi(l)} \circ \gamma_{Y',l}$.

The components of the natural transformation $\beta_0 \psi'$ are admissible epics. Define a directed diagram

$$\begin{array}{ccc} L & \xrightarrow{X'_0} & \mathcal{C} \\ l & \longmapsto & \ker(\beta_{0,\psi'(l)}) \end{array}$$

Denote by \widehat{X}'_0 the colimit of this diagram in $\mathbf{Lex}(\mathcal{C})$. The universal property of kernels induces a canonical isomorphism

$$(3.5) \quad \widehat{X}'_0 \xrightarrow{\cong} \widehat{X}_0$$

As in the proof of Theorem 3.7, X' may not be an admissible Ind-diagram. Nevertheless, let

$$\begin{array}{ccc} M & \xrightarrow{\xi} & I \\ \downarrow X_0 & \gamma_X \Rightarrow & \downarrow X_1 \\ & \mathcal{C} & \end{array}$$

be a 2-commuting triangle representing the map g_X . The isomorphism (3.5) lifts to a span of directed diagrams

$$\begin{array}{ccccc} L & \xleftarrow{\lambda} & N & \xrightarrow{\mu} & M \\ & \searrow X'_0 & \downarrow & \Rightarrow & \swarrow X_0 \\ & & \mathcal{C} & & \end{array}$$

where the maps λ and μ are final. We define a directed diagram

$$\begin{array}{ccc} N & \xrightarrow{Y_0} & \mathcal{C} \\ n & \longmapsto & X_{0,\mu(n)} \cup_{X'_{0,\lambda(n)}} Y'_{0,\lambda(n)} \end{array}$$

Denote by β_0 the natural transformation $\beta'_0 \psi \lambda$. The diagram Y_0 fits into a 2-commuting triangle

$$(3.6) \quad \begin{array}{ccccc} N & \xlongequal{\quad} & N & \xlongequal{\quad} & N \\ & \searrow X_0 \mu & \alpha_0 \Rightarrow Y_0 \xRightarrow{\beta_0} & \swarrow Z_0 \psi \lambda & \\ & & \mathcal{C} & & \end{array}$$

in which the components of α_0 are admissible monics and the components of β_0 are admissible epics. The diagram Y_0 is an admissible Ind-diagram because X_0 and Z_0 are, just as at the end of the proof of Theorem 3.7. It is of cardinality at most κ by construction.

The 2-commuting triangle 3.6 fits into 2-commuting diagram

$$\begin{array}{ccccc}
 N & \xRightarrow{\quad} & N & \xRightarrow{\quad} & N \\
 \downarrow \varphi \lambda & \searrow X_0 \mu \Rightarrow \alpha_0 & \downarrow \varphi \lambda & \searrow Y_0 \Rightarrow \beta_0 & \downarrow \varphi \lambda \\
 & \mathcal{C} & & \mathcal{C} & \\
 \downarrow \gamma_X \mu & \nearrow \gamma_Y & \downarrow \gamma_Y & \nearrow \gamma_Z \lambda & \downarrow \gamma_Z \lambda \\
 I & \xRightarrow{\quad} & I & \xRightarrow{\quad} & I \\
 & X_1 \Rightarrow \alpha_1 & & Y_1 \Rightarrow \beta_1 & \\
 & \nearrow & & \nearrow & \\
 & \mathcal{C} & & \mathcal{C} &
 \end{array}$$

This 2-commuting diagram represents the map of exact sequences of admissible Ind-objects 3.4. \square

Remark 3.13. If the category \mathcal{C} is weakly idempotent complete, we can define an inverse equivalence to 3.3 as follows. Let δ be an exact sequence

$$\hat{X} \hookrightarrow \hat{Y} \twoheadrightarrow \hat{Z}$$

of admissible Ind-objects. Define $\mathcal{EC} \downarrow^a \delta$ to be the category whose objects are admissible monics of short exact sequences in $\text{Ind}_\kappa^a(\mathcal{C})$

$$\begin{array}{ccccc}
 U & \hookrightarrow & V & \twoheadrightarrow & W \\
 \downarrow & & \downarrow & & \downarrow \\
 \hat{X} & \hookrightarrow & \hat{Y} & \twoheadrightarrow & \hat{Z}
 \end{array}$$

where the top row is an exact sequence in \mathcal{C} . Morphisms in $\mathcal{EC} \downarrow^a \delta$ correspond to morphisms of short exact sequences over δ which are admissible monics. Using that \mathcal{C} is weakly idempotent complete, we can show that the category $\mathcal{EC} \downarrow^a \delta$ is a directed poset and that δ is the colimit of the canonical functor from $\mathcal{EC} \downarrow^a \delta$ to $\mathcal{E}(\text{Lex}(\mathcal{C}))$. The inverse equivalence to 3.3 is given by

$$\begin{array}{ccccc}
 \mathcal{E}\text{Ind}_\kappa^a(\mathcal{C}) & \longrightarrow & \text{Dir}_\kappa^a(\mathcal{EC}) & \longrightarrow & \text{Ind}_\kappa^a(\mathcal{EC}) \\
 \delta \vdash & \longrightarrow & \mathcal{EC} \downarrow^a \delta & \vdash & \widehat{\mathcal{EC} \downarrow^a \delta}
 \end{array}$$

3.2.3. *Admissible Ind-Objects as a Localization.*

Proposition 3.14 (See also [21]). *Denote by $W \subset \text{Dir}_\kappa^a(\mathcal{C})$ the sub-category consisting of all morphisms of admissible Ind-diagrams given by strictly commuting triangles*

$$\begin{array}{ccc}
 I & \xrightarrow{\varphi} & J \\
 \searrow X & & \swarrow Y \\
 & \mathcal{C} &
 \end{array}$$

in which the map φ is final. The functor $\widehat{(-)}: \text{Dir}_\kappa^a(\mathcal{C}) \longrightarrow \text{Ind}_\kappa^a(\mathcal{C})$ takes morphisms in W to isomorphisms of admissible Ind-objects. The induced functor

$$\text{Dir}_\kappa^a(\mathcal{C})[W^{-1}] \longrightarrow \text{Ind}_\kappa^a(\mathcal{C})$$

is an equivalence of categories.

Proof. The functor is essentially surjective by definition. Straightening (Lemma 3.9) shows that it is full. It remains to show that it is faithful.

Suppose there exist morphisms

$$X \xrightleftharpoons[(\varphi_1, \alpha_1)]{(\varphi_0, \alpha_0)} Y$$

in $\text{Dir}_\kappa^a(\mathcal{C})$ which induce equal maps of admissible Ind-objects. For $a = 0, 1$, the pair (φ_a, α_a) induces a section of the map $I \downarrow_{\mathcal{C}} J \rightarrow I$. These sections fit into a commuting triangle

$$\begin{array}{ccc} I & \xrightleftharpoons{(\varphi_a, \alpha_a)} & I \downarrow_{\mathcal{C}} J \\ & \searrow X & \swarrow \\ & \mathcal{C} & \end{array}$$

The existence of this commuting triangle implies that, for $a = 0, 1$, the image of the map $X \xrightarrow{(\varphi_a, \alpha_a)} Y$ in the localization $\text{Dir}_\kappa^a(\mathcal{C})[W^{-1}]$ is equal to the map represented by the zig-zag

$$\begin{array}{ccccc} I & \xleftarrow{\quad} & I \downarrow_{\mathcal{C}} J & \xrightarrow{\quad} & J \\ & \searrow X & \downarrow & \Rightarrow & \swarrow Y \\ & & \mathcal{C} & & \end{array}$$

We conclude that the functor is faithful. \square

3.2.4. Functoriality of the Construction.

Proposition 3.15. *An exact functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ extends canonically to an exact functor $\tilde{F}: \text{Ind}_\kappa^a(\mathcal{C}) \longrightarrow \text{Ind}_\kappa^a(\mathcal{D})$ which fits into a 2-commuting diagram*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow & \cong & \downarrow \\ \text{Ind}_\kappa^a(\mathcal{C}) & \xrightarrow[\exists \tilde{F}]{} & \text{Ind}_\kappa^a(\mathcal{D}) \end{array}$$

If F is faithful or fully faithful, then so is \tilde{F} .

Proof. Let P be a left exact presheaf on \mathcal{C} . Recall that $\mathcal{C} \downarrow P$ denotes the category of elements of P (Definition 2.25), and that $\text{colim}_{\mathcal{C} \downarrow P} X \cong P$. The functor F induces

a colimit-preserving functor

$$\mathrm{Lex}(\mathcal{C}) \xrightarrow{\tilde{F}} \mathrm{Lex}(\mathcal{D})$$

$$P \longmapsto \mathrm{colim}_{\mathcal{C} \downarrow P} FX$$

We show that \tilde{F} preserves admissible Ind-objects. Let $\hat{X} \in \mathrm{Ind}_{\kappa}^a(\mathcal{C})$ be represented by an admissible Ind-diagram

$$I \xrightarrow{X} \mathcal{C}.$$

The functor F is exact, so

$$I \xrightarrow{FX} \mathcal{D}$$

$$i \longmapsto FX_i$$

is an admissible Ind-diagram in \mathcal{D} . The canonical map

$$\begin{array}{ccc} I & \longrightarrow & \mathcal{C} \downarrow \hat{X} \\ i & \longmapsto & (X_i \rightarrow \hat{X}) \end{array}$$

induces an isomorphism

$$\hat{X} \cong \mathrm{colim}_I X_i \xrightarrow{\cong} \mathrm{colim}_{\mathcal{C} \downarrow \hat{X}} X.$$

Similarly, we obtain an isomorphism

$$\mathrm{colim}_I FX_i \xrightarrow{\cong} \mathrm{colim}_{\mathcal{C} \downarrow \hat{X}} FX =: \tilde{F}\hat{X}.$$

This shows that $\tilde{F}\hat{X} \in \mathrm{Ind}_{\kappa}^a(\mathcal{D})$.

Now let

$$\hat{X} \hookrightarrow \hat{Y} \twoheadrightarrow \hat{Z}$$

be an exact sequence in $\mathrm{Ind}_{\kappa}^a(\mathcal{C})$. Represent this as a sequence of admissible Ind-diagrams

$$\begin{array}{ccccc} I & \longrightarrow & J & \longrightarrow & K \\ & \searrow & \downarrow & \nearrow & \\ & X & Y & Z & \\ & & \downarrow & & \\ & & \mathcal{C} & & \end{array}$$

where the components of α are admissible monics, and the components of β are admissible epics. Apply the exact functor F to obtain a sequence of admissible Ind-diagrams in \mathcal{D}

$$\begin{array}{ccccc} I & \longrightarrow & J & \longrightarrow & K \\ & \searrow & \downarrow & \nearrow & \\ & FX & FY & FZ & \\ & & \downarrow & & \\ & & \mathcal{D} & & \end{array}.$$

Taking the colimits, we obtain an exact sequence in $\mathrm{Ind}_{\kappa}^a(\mathcal{D})$ isomorphic to the sequence

$$\tilde{F}\hat{X} \longrightarrow \tilde{F}\hat{Y} \longrightarrow \tilde{F}\hat{Z}.$$

We conclude that \tilde{F} is exact.

Now suppose F is faithful. For any $X: I \longrightarrow \mathcal{C}$ and $Y: J \longrightarrow \mathcal{C}$ in $\text{Dir}_\kappa^a(\mathcal{C})$, we have

$$\text{hom}_{\text{Ind}_\kappa^a(\mathcal{C})}(\hat{X}, \hat{Y}) \cong \lim_I \text{colim}_J \text{hom}_{\mathcal{C}}(X_i, Y_j).$$

The construction of directed colimits and inductive limits in the category of sets shows that a map of directed or inductive diagrams which is injective at each object in the diagram induces an injection in the colimit or limit. If F is faithful, then

$$\begin{aligned} \text{hom}_{\text{Ind}_\kappa^a(\mathcal{C})}(\hat{X}, \hat{Y}) &\cong \lim_I \text{colim}_J \text{hom}_{\mathcal{C}}(X_i, Y_j) \\ &\subseteq \lim_I \text{colim}_J \text{hom}_{\mathcal{D}}(FX_i, FY_j) \\ &\cong \text{hom}_{\text{Ind}_\kappa^a(\mathcal{D})}(\tilde{F}\hat{X}, \tilde{F}\hat{Y}) \end{aligned}$$

and we conclude that \tilde{F} is faithful as well. If F is fully faithful, the previous argument shows \tilde{F} is as well. \square

3.2.5. Countable Admissible Ind-Objects. Countable admissible Ind-objects have been studied in the literature for many years. We take Keller [14] as a basic reference.

Definition 3.16. [14, Appendix B] The *countable envelope* \mathcal{C}^\sim of an exact category \mathcal{C} is the full sub-category of $\text{Lex}(\mathcal{C})$ consisting of all left-exact presheaves \hat{X} which are representable by an admissible Ind-diagram $X: \mathbb{N} \longrightarrow \mathcal{C}$.

Proposition 3.17. *The embedding $\mathcal{C}^\sim \hookrightarrow \text{Ind}_{\aleph_0}^a(\mathcal{C})$ is an equivalence of exact categories.*

Proof. The embedding is fully faithful by definition. We show it is essentially surjective. Let $\hat{X} \in \text{Ind}_{\aleph_0}^a(\mathcal{C})$ be represented by a countable admissible Ind-diagram $X: I \longrightarrow \mathcal{C}$. Every countable directed poset I admits a final map $f: \mathbb{N} \longrightarrow I$.⁸ The isomorphism $\text{colim}_{n \in \mathbb{N}} X_{f(n)} \cong \hat{X}$ shows that $\hat{X} \in \mathcal{C}^\sim$. \square

Recall that an exact category \mathcal{C} is *split exact* if every exact sequence in \mathcal{C} splits.

Proposition 3.18. *Let \mathcal{C} be a split exact category. Then $\text{Ind}_{\aleph_0}^a(\mathcal{C})$ is split exact.*

Proof. By Propositions 3.12 and 3.17, it suffices to show that every exact sequence of countable admissible Ind-diagrams

$$\begin{array}{ccccc} \mathbb{N} & \xlongequal{\quad} & \mathbb{N} & \xlongequal{\quad} & \mathbb{N} \\ & \searrow & \downarrow & \swarrow & \\ & X & Y & Z & \\ & & \downarrow & & \\ & & \mathcal{C} & & \end{array}$$

splits. By the usual argument, it suffices to construct a splitting of α . Denote by

$$\alpha_{\leq n}: X_{\leq n} \longrightarrow Y_{\leq n}$$

the restriction of α to $\{0 < \dots < n\} \subset \mathbb{N}$. We induct on n to show that a retract of $\alpha_{\leq n}$ exists for all n .

⁸Pick a bijection $\mathbb{N} \rightarrow I$. We construct f by induction. Let $f(0) := i_0$. Suppose we have defined $f(l)$ for $0 \leq l \leq n$. Pick $f(n+1) \in I$ such that $f(n+1) \geq i_l$ for $0 \leq l \leq n$ and such that $f(n+1) \geq f(n)$. This completes the induction step.

Because \mathcal{C} is split exact, a retract of $\alpha_0: X_0 \twoheadrightarrow Y_0$ exists. Now suppose that a retract $\rho_{\leq n}$ of $\alpha_{\leq n}$ exists. It suffices to construct a retract ρ_{n+1} of α_{n+1} which fits into a commuting square

$$\begin{array}{ccc} X_n & \longrightarrow & X_{n+1} \\ \uparrow \rho_n & & \uparrow \rho_{n+1} \\ Y_n & \longrightarrow & Y_{n+1} \end{array}$$

The maps ρ_n and α_n induce an isomorphism $Y_n \cong X_n \oplus B_0$. We can also choose splittings $X_{n+1} \cong X_n \oplus A$ and $Y_{n+1} \cong Y_n \oplus B_1$. With respect to these splittings, we can write α_{n+1} as the map $(1_{X_n} + \chi, \sigma_0, \sigma_1)$ where $\chi: A \rightarrow X_n$, and $\sigma_i: A \rightarrow B_i$ for $i = 0, 1$.

The map $(\sigma_0, \sigma_1): A \rightarrow B_0 \oplus B_1$ is an admissible monic, as it is isomorphic to the pushout of α_{n+1} along the projection $\pi_A: X_{n+1} \rightarrow A$. Choose a retraction ρ' of (σ_0, σ_1) , and define $\rho_{n+1} := \pi_{X_n} \oplus \rho' - \chi \rho' \pi_{B_0 \oplus B_1}$, where $\pi_{(-)}$ denotes the projection onto $(-)$. Then ρ_{n+1} is a retraction of α_{n+1} and fits into the commuting square above. This completes the induction. \square

Proposition 3.19. *Let \mathcal{C} be a split exact category for which there exists a collection of objects $\{S_i\}_{i \in \mathbb{N}} \subset \mathcal{C}$ such that every object $Y \in \mathcal{C}$ is a direct summand of $\bigoplus_{i=0}^n S_i$ for some n . Then every countable Ind-object in \mathcal{C} is a direct summand of*

$$\widehat{\bigoplus_{\mathbb{N}} S} := \operatorname{colim}_{(m,n) \in \mathbb{N} \times \mathbb{N}} \bigoplus_{i=0}^m S_i^{\oplus n}.$$

Proof. Let $X: \mathbb{N} \rightarrow \mathcal{C}$ be an admissible Ind-diagram representing a countable Ind-object $\hat{X} \in \operatorname{Ind}_{\mathbb{N}_0}^a(\mathcal{C})$. We construct, by induction, an admissible diagram $X': \mathbb{N} \rightarrow \mathcal{C}$ and an isomorphism of $X \oplus X'$ with a final sub-diagram of

$$\begin{aligned} \bigoplus_{\mathbb{N}} S: \mathbb{N} \times \mathbb{N} &\longrightarrow \mathcal{C} \\ (m, n) &\mapsto \bigoplus_{i=0}^m S_i^{\oplus n} \end{aligned}$$

For $n = 0$, there exists, by assumption, an object $X'_0 \in \mathcal{C}$, a number $m_0 \in \mathbb{N}$, and an isomorphism $X_0 \oplus X'_0 \cong \bigoplus_{i=0}^{m_0} S_i$.

For the induction step, assume we have constructed an admissible diagram $X': \mathbb{N}_{\leq k} \rightarrow \mathcal{C}$, a sequence $\{m_0 < \dots < m_k\} \in \mathbb{N}$ and a natural collection of isomorphisms $X_j \oplus X'_j \xrightarrow[\rho_j]{\cong} \bigoplus_{i=0}^{m_j} S_i^{\oplus j+1}$.

Because \mathcal{C} is split exact, the admissible monic $X_k \hookrightarrow X_{k+1}$ is isomorphic to the inclusion of the direct summand $X_k \hookrightarrow X_k \oplus A$ for some $A \in \mathcal{C}$. By assumption, there exists an object $A' \in \mathcal{C}$, a number $m_{k+1} \in \mathbb{N}$ and an isomorphism

$$A \oplus A' \xrightarrow[\alpha]{\cong} \bigoplus_{i=0}^{m_{k+1}} S_i,$$

with $m_{k+1} > m_k$. Define $X'_{k+1} := X'_k \oplus \bigoplus_{i=m_k+1}^{m_{k+1}} S_i^{\oplus k+1} \oplus A'$. The inclusion of the X'_k -summand defines an admissible monic $X'_k \hookrightarrow X'_{k+1}$, and thus an extension

of the admissible diagram X' to $\mathbb{N}_{\leq k+1}$. Define the isomorphism ρ_{k+1} to be the composite

$$\begin{aligned} X_{k+1} \oplus X'_{k+1} &\xrightarrow{\cong} X_k \oplus X'_k \oplus \bigoplus_{i=m_k+1}^{m_{k+1}} S_i^{\oplus k+1} \oplus A \oplus A' \\ &\xrightarrow[\rho_k \oplus 1 \oplus \alpha]{\cong} \bigoplus_{i=0}^{m_k} S_i^{\oplus k+1} \oplus \bigoplus_{i=m_k+1}^{m_{k+1}} S_i^{\oplus k+1} \oplus \bigoplus_{i=0}^{m_{k+1}} S_i \xrightarrow{\cong} \bigoplus_{i=0}^{m_{k+1}} S_i^{\oplus k+2} \end{aligned}$$

By construction, the collection of isomorphisms $\{\rho_j\}_{j=0}^{k+1}$ is a natural with respect to the maps in the admissible diagrams X , X' and $j \mapsto \bigoplus_{i=0}^{m_j} S_i^{\oplus j+1}$. This completes the induction step.

We conclude that there exists an admissible diagram $X': \mathbb{N} \rightarrow \mathcal{C}$, an increasing sequence $\{m_j\}_{j \in \mathbb{N}} \subset \mathbb{N}$ and a natural isomorphism from $X \oplus X'$ to the admissible diagram $j \mapsto \bigoplus_{i=0}^{m_j} S_i^{\oplus j+1}$. Because this last diagram includes as a final sub-diagram of the diagram $\bigoplus_{\mathbb{N}} S$, we conclude that $\widehat{X} \oplus \widehat{X'} \cong \widehat{\bigoplus_{\mathbb{N}} S}$. \square

Example 3.20. Let R be a ring. The category $\text{Ind}_{\aleph_0}^a(P^f(R))$ is split exact, and every countable admissible Ind-module is a direct summand of $R[t]$.

Remark 3.21. The inductions in the previous proofs do not have an obvious analogue for uncountable Ind-diagrams, primarily because, unlike countable ones, they do not in general contain a linearly ordered final sub-diagram. At present, we do not know whether the result fails, or whether one merely needs a different proof.

3.2.6. $\text{Ind}_{\kappa}^a(\mathcal{C})$ is Not Generally Idempotent Complete. In this short subsection we show that there exists an idempotent complete category \mathcal{C} , such that $\text{Ind}_{\kappa}^a(\mathcal{C})$ is not idempotent complete. Note that Freyd [11] has shown that an additive category which admits infinite direct sums, such as $\text{Ind}_{\kappa}^a(\mathcal{C})$, is weakly idempotent complete if and only if it is idempotent complete. This example shows that, in general, neither condition on \mathcal{C} is inherited by $\text{Ind}_{\kappa}^a(\mathcal{C})$.

Recall that a finitely presented module M over a ring R is *perfect* if it admits a projective resolution of finite length. Denote the category of perfect modules by $\text{perf}(R)$. This category is closed under extensions in $\text{Mod}(R)$, and is also idempotent complete. A K -theoretic construction allows us to verify that for $\mathcal{C} = \text{perf}(R[t])$, where R is a carefully chosen ring, $\text{Ind}_{\kappa}^a(\mathcal{C})$ is not idempotent complete.

Let k be a field, and consider the k -algebras $k[t]$ and $k[t, t^{-1}]$. Denote by $\text{Mod}^f(k[t])$ the category of finitely presented $k[t]$ -modules, and likewise for $k[t, t^{-1}]$. The $k[t]$ -algebra structure on $k[t, t^{-1}]$ gives rise to a functor

$$\text{Mod}^f(k[t, t^{-1}]) \longrightarrow \text{Mod}(k[t]).$$

Since every $k[t]$ -module is the direct limit of its finitely presented sub-modules, we obtain a natural functor

$$\mathbf{j}_*: \text{Mod}^f(k[t, t^{-1}]) \longrightarrow \text{Ind}_{\kappa}^a(\text{Mod}^f(k[t])).$$

For example, we see that

$$k[t, t^{-1}] = \bigcup_{n \in \mathbb{N}} t^{-n} k[t].$$

Since $k[t]$ is a principal ideal domain, the classification theorem (e.g. [31, 4.2.10 and 4.2.11]) implies that every finitely generated module has a projective resolution

of length two, i.e. $\text{Mod}^f(k[t]) \simeq \text{perf}(k[t])$. By restricting \mathbf{j}_* to $P^f(-) \subset \text{Mod}^f(-)$, we obtain

$$\mathbf{j}_*: P^f(k[t, t^{-1}]) \longrightarrow \text{Ind}_\kappa^a(\text{perf}(k[t])).$$

Let A be a k -algebra. Analogously to \mathbf{j}_* , we will construct an exact functor

$$\Phi: P^f(A[t, t^{-1}]) \longrightarrow (\text{Ind}_\kappa^a(\text{perf}(A[t])))^{ic}.$$

Our strategy will be to construct a functor

$$\varphi: F^f(A[t, t^{-1}]) \longrightarrow \text{Ind}_\kappa^a(\text{perf}(A[t])).$$

Because $P^f(-) \simeq F^f(-)^{ic}$, the universal property of idempotent completions guarantees that φ will extend uniquely to the desired exact functor. We show $\text{Ind}_\kappa^a(\text{perf}(A[t]))$ is not idempotent complete by showing that Φ does not factor through the inclusion $\text{Ind}_\kappa^a(\text{perf}(A[t])) \hookrightarrow \text{Ind}_\kappa^a(\text{perf}(A[t]))^{ic}$.

We now define φ . Using the isomorphism $A[t, t^{-1}]^n \cong A \otimes_k k[t, t^{-1}]^n$, we define

$$\varphi(A[t, t^{-1}]^n) := A \otimes_k \mathbf{j}_*(k[t, t^{-1}]^n).$$

Note that $A \otimes_k \mathbf{j}_*(k[t, t^{-1}]^n)$ is well-defined as an object of $\text{Ind}_\kappa^a(\text{perf}(A[t]))$, because the functor $A \otimes_k -$ preserves perfect modules.

Now let A be a k -algebra with non-vanishing $K_{-1}(A)$. According to [32, Def. III.4.1] we have

$$K_{-1}(A) \cong \text{coker}(K_0(A[t]) \oplus K_0(A[t^{-1}]) \xrightarrow{\pm} K_0(A[t, t^{-1}])).$$

Therefore, our assumption on A implies that the morphism $K_0(A[t]) \longrightarrow K_0(A[t, t^{-1}])$ is not surjective. In particular, there exists a class in $K_0(A[t, t^{-1}])$, represented by a finitely generated projective module M , which is not of the form $A[t, t^{-1}] \otimes_{A[t]} N$ with $N \in P^f(A[t])$.

Suppose $M \in P^f(A[t, t^{-1}])$ is such a module. Above we have shown how to construct the element $\Phi(M)$ of $(\text{Ind}_\kappa^a(\text{perf}(A[t])))^{ic}$. If $\Phi(M)$ is an element of $\text{Ind}_\kappa^a(\text{perf}(A[t]))$, then $\Phi(M)$ can be expressed as an increasing union of perfect $A[t]$ -modules $\{M_i\}_{i \in I}$. In particular, since M is finitely generated over $A[t, t^{-1}]$, there would be an index i_0 , such that $A[t, t^{-1}] \otimes_{A[t]} M_{i_0} \cong M$. But this contradicts our assumption on M . We therefore conclude that $\text{Ind}_\kappa^a(\text{perf}(A[t]))$ is not idempotent complete.

3.3. Examples. Let R be a ring. In this section, we describe the categories of admissible Ind-objects in various categories of finitely generated R -modules.

Example 3.22.

- (1) Let R be a Noetherian ring. Denote by $\text{Mod}^f(R)$ the abelian category of finitely presented R -modules. Every R -module is the directed colimit of its finitely presented sub-modules. As a consequence, the category $\text{Ind}_\kappa^a(\text{Mod}^f(R))$ is equivalent to the abelian category of R -modules having at most κ generators. If we omit the cardinality bound, we have $\text{Ind}^a(\text{Mod}^f(R)) \simeq \text{Mod}(R)$.
- (2) Similarly, if X is a Noetherian scheme, $\text{Ind}^a(\text{Coh}(X)) \simeq \text{QCoh}(X)$.

Example 3.23. Recall that $F^f(R)$ denotes the category of finitely generated free modules. The category $\text{Ind}_{\aleph_0}^a(F^f(R))$ consists of countably generated free modules. In particular, up to isomorphism, it contains objects R^n for $n \in \mathbb{N}$ and the module $R[t]$, along with all R -linear maps between these.

Proposition 3.24. *The category $P(R)$ of projective R -modules embeds fully faithfully into the idempotent completion of $\mathrm{Ind}^a(P^f(R))$. This embedding restricts to an equivalence between the category $P^{\aleph_0}(R)$ of countably generated projective R -modules and the idempotent completion of $\mathrm{Ind}_{\aleph_0}^a(P^f(R))$.*

Proof. Every projective R -module is the image of an idempotent on a free R -module, so it is enough to show that the category of free modules is a full sub-category of $\mathrm{Ind}^a(P^f(R))$.

Every set is the directed colimit of its finite sub-sets, and the free module functor $F: \mathbf{Set} \rightarrow \mathbf{Mod}(R)$ preserves colimits. Further, the free functor takes directed diagrams of injections of finite sets to admissible Ind-diagrams in $P^f(R)$. We conclude that any free module M is the direct colimit of the admissible Ind-diagram of sub-modules generated by finite sub-sets of a given basis. This shows that free modules form a sub-category of $\mathrm{Ind}^a(P^f(R))$. This sub-category is full because both free modules and admissible Ind-modules are full sub-categories of $\mathbf{Mod}(R)$.

Proposition 3.19 shows that the restriction of this embedding to $P^{\aleph_0}(R)$ is essentially surjective. \square

Definition 3.25. A *flat Mittag-Leffler module* over R is a (left) module M which is isomorphic to the colimit of a directed diagram $M: I \rightarrow P^f(R)$ such that for every $i \in I$ there exists $j \geq i$, with

$$\mathrm{Im}(P_k^\vee \rightarrow P_i^\vee) = \mathrm{Im}(P_j^\vee \rightarrow P_i^\vee)$$

for all $k \geq j$.

Proposition 3.26. *The category $\mathrm{Ind}^a(P^f(R))$ embeds as a full sub-category of the category of flat Mittag-Leffler R -modules.*

Proof. The equivalence $\mathbf{Lex}(P^f(R)) \simeq \mathbf{Mod}(R)$ embeds $\mathrm{Ind}^a(P^f(R))$ as a full sub-category of $\mathbf{Mod}(R)$. We show that $\mathrm{Ind}^a(P^f(R))$ is a full sub-category of the category of flat Mittag-Leffler modules. Indeed, an admissible monic $N_i \hookrightarrow N_j$ in $P^f(R)$ is an inclusion of a direct summand. In particular, $N_i \hookrightarrow N_j$ induces a surjective map $N_j^\vee \twoheadrightarrow N_i^\vee$. We conclude that the colimit in $\mathbf{Mod}(R)$ of any admissible Ind-diagram in $P^f(R)$ is a flat Mittag-Leffler module. \square

Remark 3.27. Direct summands of flat Mittag-Leffler modules are also flat Mittag-Leffler. As a result, the discussion in Section 3.2.6 shows that $\mathrm{Ind}^a(P^f(R))$ is not equivalent, in general, to the category of flat Mittag-Leffler modules.

Theorem 3.28 (Kaplansky–Raynaud–Gruson⁹). *An R -module M is projective if and only if M is a direct sum of countably generated R -modules and M is a flat Mittag-Leffler module.*

As a consequence of this theorem and Proposition 3.24, we have the following.

Corollary 3.29. *The sub-category $P(R) \subset \mathrm{Ind}^a(P^f(R))^{ic}$ is the full sub-category consisting of arbitrary direct sums of objects in $\mathrm{Ind}_{\aleph_0}^a(P^f(R))^{ic}$.*

⁹See [10, Theorem 2.2].

3.4. The Calkin Category. Let R be a ring. The Calkin algebra $\text{Calk}(R)$ is the quotient of the algebra $\text{End}_R(R[t])$ by the ideal of finite rank endomorphisms. By analogy, we define the following.

Definition 3.30. Let \mathcal{C} be an exact category. Define the κ -Calkin category by

$$\text{Calk}_\kappa(\mathcal{C}) := (\text{Ind}_\kappa^a(\mathcal{C})/\mathcal{C})^{ic}$$

Example 3.31. Let R be a local ring. Then $\text{Calk}_{\aleph_0}(P^f(R))$ has two objects up to isomorphism, the zero object and $R[t]$.¹⁰ By construction,

$$\text{End}_{\text{Calk}_{\aleph_0}(P^f(R))}(R[t]) \cong \text{Calk}(R).$$

Remark 3.32. One feature of the Calkin algebra $\text{Calk}(R)$ is the isomorphism

$$K_i(\text{Calk}(R)) \cong K_{i-1}(R)$$

in K -theory. Similarly, Schlichting [27] has shown that

$$K_i(\text{Calk}_\kappa(\mathcal{C})) \cong K_{i-1}(\mathcal{C})$$

when \mathcal{C} is idempotent complete.

4. ADMISSIBLE PRO-OBJECTS

Admissible Pro-objects are dual to admissible Ind-objects. They sit in relation to objects of \mathcal{C} as the topological R -module $R[[t]]$, with the t -adic topology, sits in relation to finitely generated free R -modules. More generally, the results of this section form an elaboration, in the setting of exact categories, of Artin–Mazur [2, Appendix 2].

4.1. The Category of Admissible Pro-Objects and its Properties.

Definition 4.1. Let \mathcal{C} be an exact category. Let κ be an infinite cardinal. The category $\text{Pro}_\kappa^a(\mathcal{C})$ of *admissible Pro-objects in \mathcal{C} of size at most κ* is the opposite of the category $\text{Ind}_\kappa^a(\mathcal{C}^{op})$, i.e.

$$\text{Pro}_\kappa^a(\mathcal{C}) := \text{Ind}_\kappa^a(\mathcal{C}^{op})^{op}.$$

We can also omit the cardinality bound and define

$$\text{Pro}^a(\mathcal{C}) := \text{Ind}^a(\mathcal{C}^{op})^{op}.$$

The proofs in the previous section dualize to give the following.

Theorem 4.2.

- (1) *The category $\text{Pro}_\kappa^a(\mathcal{C})$ is closed under extensions in $\text{Lex}(\mathcal{C}^{op})^{op}$. If \mathcal{C} is weakly idempotent complete, then $\text{Pro}_\kappa^a(\mathcal{C})$ is right special in $\text{Lex}(\mathcal{C}^{op})^{op}$.*
- (2) *An exact category \mathcal{C} embeds as a right s -filtering sub-category of $\text{Pro}_\kappa^a(\mathcal{C})$.*
- (3) *The category $\text{Pro}_\kappa^a(\mathcal{EC})$ is canonically equivalent to $\mathcal{E}\text{Pro}_\kappa^a(\mathcal{C})$.*
- (4) *Define the category $\text{Inv}_\kappa^a(\mathcal{C})$ of admissible Pro-diagrams by*

$$\text{Inv}_\kappa^a(\mathcal{C}) := \text{Dir}_\kappa^a(\mathcal{C}^{op})^{op}.$$

The category $\text{Pro}_\kappa^a(\mathcal{C})$ is the localization of $\text{Inv}_\kappa^a(\mathcal{C})$ at the sub-category of co-final morphisms.

¹⁰Because R is local, all projective modules are free, so Corollary 3.29 shows that $R[t]$ is the only object in $\text{Ind}_{\aleph_0}^a(P^f(R)) \setminus P^f(R)$.

- (5) An exact functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ extends canonically to an exact functor

$$\mathrm{Pro}_\kappa^a(\mathcal{C}) \xrightarrow{\tilde{F}} \mathrm{Pro}_\kappa^a(\mathcal{D})$$

If F is faithful or fully faithful, then the same is true of \tilde{F} .

- (6) The category $\mathrm{Pro}_{\aleph_0}^a(\mathcal{C})$ is split exact if \mathcal{C} is.
 (7) If \mathcal{C} is split exact and there exists a collection of objects $\{S_i\}_{i \in \mathbb{N}} \subset \mathcal{C}$ such that every object $Y \in \mathcal{C}$ is a direct summand of $\bigoplus_{i=0}^n S_i$ for some n . Then every countable Pro-object in \mathcal{C} is a direct summand of

$$\widehat{\prod_{\mathbb{N}} S} := \lim_{(m,n) \in \mathbb{N} \times \mathbb{N}} \prod_{i=0}^m S_i^{\times n}.$$

- (8) The category $\mathrm{Pro}_\kappa^a(\mathcal{C})$ is not in general idempotent complete.

4.2. Examples. Let R be a ring. We now relate admissible Pro-objects in $P^f(R)$ to categories of topological R -modules.

Definition 4.3. Let R be a ring, and let M be an R -module. Endow both R and M with the discrete topology. The *topological dual* M^\vee is the module $\mathrm{hom}_R(M, R)$ with topology induced by the product topology on R^M . Concretely, denote by $N \subset M$ a finitely generated sub-module. The sub-modules

$$U_N := \{f \in M^\vee \mid f(x) \neq 0 \text{ for all } x \in N\}$$

form a basis of open neighborhoods of M^\vee .

Proposition 4.4. Denote by $P^f(R^\circ)$ the category of finitely generated projective (right) R -modules. Endow R and objects of $P^f(R^\circ)$ with the discrete topology. The topological dual gives an equivalence of categories

$$\mathrm{Ind}_\kappa^a(P^f(R^\circ))^{op} \xrightarrow{(-)^\vee} \mathrm{Pro}_\kappa^a(P^f(R)).$$

Proof. Denote by $\mathrm{Mod}(R)_{top}$ the category of topological (left) R -modules and continuous maps. Denote by $d : \mathrm{Mod}(R)^{op} \hookrightarrow \mathrm{Mod}(R)_{top}^{op}$ the embedding given by considering abstract modules as discrete modules. The assignment

$$\widehat{M} \mapsto \lim_{(\widehat{M} \downarrow (P^f(R))^{op})^{op}} d(P)$$

extends to a faithful embedding $\tau : \mathrm{Pro}_\kappa^a(P^f(R)) \hookrightarrow \mathrm{Mod}(R)_{top}$.

This embedding is full. Indeed, the straightening argument applies in $\mathrm{Mod}(R)_{top}^{op}$ to show that if

$$I \xrightarrow{P} \mathrm{Mod}(R)_{top}^{op} \quad \text{and} \quad J \xrightarrow{Q} \mathrm{Mod}(R)_{top}^{op}$$

are directed diagrams of discrete R -modules in which the morphisms are injective, then any map

$$\mathrm{colim}_I P \longrightarrow \mathrm{colim}_J Q$$

in $\text{Mod}(R)_{top}^{op}$ can be represented by a span of directed diagrams of discrete R -modules

$$\begin{array}{ccccc} I & \xleftarrow{\varphi} & I & \downarrow \text{Mod}(R)^{op} & J \xrightarrow{\psi} J \\ & \searrow P & & \Rightarrow & \swarrow Q \\ & & \text{Mod}(R)^{op} & & \end{array}$$

where φ and ψ are final. In particular, any continuous map $\tau(\widehat{M}) \rightarrow \tau(\widehat{N})$ is the limit of such a span, and is therefore in the image of the embedding τ .

We show that the topological dual of $\widehat{X} \in \text{Ind}_\kappa^a(P^f(R^\circ))$ is in the image of τ . Represent \widehat{X} by an admissible Ind-diagram $X: I \rightarrow P^f(R^\circ)$. Then

$$\begin{aligned} \widehat{X}^\vee &\cong (\text{colim}_I X_i)^\vee \\ &\cong \lim_I X_i^\vee. \end{aligned}$$

The diagram $X^\vee: I \rightarrow P^f(R)^{op}$ is an admissible Pro-diagram in $P^f(R)$, because the topological dual takes admissible monics of finitely generated projective (right) modules to admissible epics of finitely generated projective (left) modules.

We have shown that the topological dual induces a fully faithful embedding

$$(-)^\vee: \text{Ind}_\kappa^a(P^f(R^\circ))^{op} \hookrightarrow \text{Pro}_\kappa^a(P^f(R)).$$

That this is an equivalence follows from the fact that

$$(-)^\vee: P^f(R)^{op} \longrightarrow P^f(R^\circ)$$

is an equivalence. \square

Corollary 4.5. *An admissible Ind-module is the topological dual of its topological dual, and similarly for an admissible Pro-module.*

Corollary 4.6. *Denote by $P^\kappa(R^\circ)$ the category of projective (right) R -modules with generating sets of cardinality less than or equal to κ . The topological dual gives a fully faithful embedding*

$$P^\kappa(R^\circ)^{op} \xhookrightarrow{(-)^\vee} (\text{Pro}_\kappa^a(P^f(R)))^{ic}$$

This embedding restricts to an equivalence of categories

$$P^{\aleph_0}(R^\circ)^{op} \simeq \text{Pro}_{\aleph_0}^a(P^f(R))^{ic}.$$

Proof. The topological dual induces an equivalence of categories

$$(\text{Pro}^a(P^f(R)))^{ic} \simeq ((\text{Ind}^a(P^f(R^\circ)))^{ic})^{op}.$$

The category $P(R^\circ)^{op}$ embeds as a full sub-category of $((\text{Ind}^a(P^f(R^\circ)))^{ic})^{op}$ by Proposition 3.24. \square

5. TATE OBJECTS

We are now ready to introduce the category $\text{Tate}^{el}(\mathcal{C})$ of *elementary Tate objects* and its idempotent completion $\text{Tate}(\mathcal{C})$. Elementary Tate objects sit in relation to objects of \mathcal{C} as the topological R -module $R((t))$ sits in relation to finitely generated free R -modules.

5.1. The Category of Elementary Tate Objects.

Definition 5.1. Let \mathcal{C} be an exact category and let κ be an infinite cardinal. An *admissible Ind-Pro object in \mathcal{C} of size at most κ* is an object in the category $\text{Ind}_\kappa^a(\text{Pro}_\kappa^a(\mathcal{C}))$.

Definition 5.2. Let \mathcal{C} be an exact category. An *elementary Tate diagram in \mathcal{C} of size at most κ* is an admissible Ind-diagram

$$I \xrightarrow{X} \text{Pro}_\kappa^a(\mathcal{C})$$

of cardinality at most κ such that, for all $i \leq i'$ in I , the object $X_{i'}/X_i$ is in \mathcal{C} . Denote by $\mathbb{T}_\kappa(\mathcal{C}) \subset \text{Dir}_\kappa^a(\text{Pro}_\kappa^a(\mathcal{C}))$ the category of elementary Tate diagrams in \mathcal{C} of size at most κ .

By definition, we have a canonical functor

$$(5.1) \quad \widehat{(-)}: \mathbb{T}_\kappa(\mathcal{C}) \longrightarrow \text{Ind}_\kappa^a(\text{Pro}_\kappa^a(\mathcal{C})).$$

Definition 5.3. Define the category $\text{Tate}_\kappa^{el}(\mathcal{C})$ of *elementary Tate objects in \mathcal{C} of size at most κ* to be the full sub-category of $\text{Ind}_\kappa^a(\text{Pro}_\kappa^a(\mathcal{C}))$ consisting of objects in the essential image of (5.1). Denote by $\text{Tate}^{el}(\mathcal{C})$ the analogous full sub-category of $\text{Ind}^a(\text{Pro}^a(\mathcal{C}))$.

Theorem 5.4. Let \mathcal{C} be an exact category. Let κ be an infinite cardinal. The category $\text{Tate}_\kappa^{el}(\mathcal{C})$ is the smallest full sub-category of $\text{Ind}_\kappa^a(\text{Pro}_\kappa^a(\mathcal{C}))$ which

- (1) contains the sub-category $\text{Ind}_\kappa^a(\mathcal{C}) \subset \text{Ind}_\kappa^a(\text{Pro}_\kappa^a(\mathcal{C}))$,
- (2) contains the sub-category $\text{Pro}_\kappa^a(\mathcal{C}) \subset \text{Ind}_\kappa^a(\text{Pro}_\kappa^a(\mathcal{C}))$, and
- (3) is closed under extensions.

In particular, $\text{Tate}_\kappa^{el}(\mathcal{C})$ admits a canonical structure as an exact category.

The theorem allows us to quickly produce examples of elementary Tate objects.

Example 5.5. Let X be an integral curve over a field k . Denote the set of closed points by $|X|$. For each closed point $x \in |X|$, let $\mathcal{O}_{X,x}$ denote the local ring at x , let $\widehat{\mathcal{O}_{X,x}}$ denote its completion with respect to the maximal ideal, and let $\text{Frac}(\widehat{\mathcal{O}_{X,x}})$ denote the field of fractions of the completed local ring. The *ring of adèles* $\mathbb{A}(X)$ is the restricted product

$$\mathbb{A}(X) := \prod'_{x \in |X|} \text{Frac}(\widehat{\mathcal{O}_{X,x}})$$

where, for any $f \in \mathbb{A}(X)$, the factor $f(x)$ lies in $\widehat{\mathcal{O}_{X,x}}$ for all but finitely many $x \in |X|$.

If the set of closed points of X has cardinality κ , then $\mathbb{A}(X)$ is an elementary Tate vector space over k of size κ . Indeed, $\mathbb{A}(X)$ is isomorphic as a k -vector space to the direct sum

$$\prod_{x \in |X|} \widehat{\mathcal{O}_{X,x}} \oplus \bigoplus_{x \in |X|} \text{Frac}(\widehat{\mathcal{O}_{X,x}})/\widehat{\mathcal{O}_{X,x}}$$

The product $\prod_{x \in |X|} \widehat{\mathcal{O}_{X,x}}$ is an admissible Pro-vector space of size κ , while the coproduct $\bigoplus_{x \in |X|} \text{Frac}(\widehat{\mathcal{O}_{X,x}})/\widehat{\mathcal{O}_{X,x}}$ is an admissible Ind-vector space of size κ . The category $\text{Tate}_\kappa^{el}(\text{Vect}_k)$ is closed under extensions in $\text{Ind}_\kappa^a(\text{Pro}_\kappa^a(\text{Vect}_k))$, so $\mathbb{A}(X)$ is an object in $\text{Tate}_\kappa^{el}(\text{Vect}_k)$.

Proof of Theorem 5.4. The definition of an elementary Tate object immediately implies that the embeddings $\text{Ind}_\kappa^a(\mathcal{C}) \hookrightarrow \text{Ind}_\kappa^a(\text{Pro}_\kappa^a(\mathcal{C}))$ and $\text{Pro}_\kappa^a(\mathcal{C}) \hookrightarrow \text{Ind}_\kappa^a(\text{Pro}_\kappa^a(\mathcal{C}))$ factor through the inclusion $\text{Tate}_\kappa^{\text{el}}(\mathcal{C}) \subset \text{Ind}_\kappa^a(\text{Pro}_\kappa^a(\mathcal{C}))$. We show that the subcategory $\text{Tate}_\kappa^{\text{el}}(\mathcal{C})$ is closed under extensions, and that every elementary Tate object arises as an extension of an admissible Ind-object by an admissible Pro-object.

Let

$$(5.2) \quad \widehat{X} \hookrightarrow F \twoheadrightarrow \widehat{Z}$$

be an exact sequence of admissible Ind-Pro objects such that \widehat{X} and \widehat{Z} are elementary Tate objects.

Observe that for any elementary Tate diagram

$$I \xrightarrow{W} \text{Pro}_\kappa^a(\mathcal{C})$$

and any final map $J \rightarrow I$, the restriction of W to J is also an elementary Tate diagram. Lemma 3.8 and the proof of Theorem 3.7 imply that we can lift the short exact sequence 5.2 to a sequence of admissible Ind-diagrams of admissible Pro-objects

$$\begin{array}{ccccc} I & \longrightarrow & I & \longrightarrow & I \\ & \searrow & \downarrow \alpha & \searrow & \\ & & F & & \\ & \swarrow X & \downarrow \beta & \swarrow Z & \\ & & \mathcal{C} & & \end{array}$$

such that X and Z are elementary Tate diagrams of size at most κ , such that the components of β are admissible epics, and such that the components of α are admissible monics.

For each $i \leq j$ in I , we have a commuting diagram of admissible Pro-objects with exact rows

$$\begin{array}{ccccc} X_i & \hookrightarrow & F_i & \twoheadrightarrow & Z_i \\ \downarrow & & \downarrow & & \downarrow \\ X_j & \hookrightarrow & F_j & \twoheadrightarrow & Z_j \end{array}$$

All vertical maps are admissible monics in $\text{Pro}_\kappa^a(\mathcal{C})$. The (3×3) -Lemma [9, Corollary 3.6] shows that taking the cokernels of the vertical maps gives an exact sequence of admissible Pro-objects

$$X_j/X_i \hookrightarrow F_j/F_i \twoheadrightarrow Z_j/Z_i$$

The first and last terms are in \mathcal{C} , and \mathcal{C} is closed under extensions in $\text{Pro}_\kappa^a(\mathcal{C})$ (Theorem 4.2). We conclude that F is an elementary Tate diagram.

It remains to show that every elementary Tate object is an extension of an admissible Ind-object by an admissible Pro-object. Lemma 3.11 shows that, given an elementary Tate diagram $X: I \rightarrow \text{Pro}_\kappa^a(\mathcal{C})$, for any $i \in I$, the map $X_i \hookrightarrow \widehat{X}$ is an admissible monic. The sub-object X_i is an admissible Pro-object, and, because X_j/X_i is in \mathcal{C} for all $j \geq i$, the quotient \widehat{X}/X_i is an admissible Ind-object. \square

5.2. Properties of Elementary Tate Objects. The properties of admissible Ind-objects established in Section 3.2 have their counterparts for elementary Tate objects. We develop these here.

5.2.1. $\text{Pro}_\kappa^a(\mathcal{C})$ as an Exact, Full Sub-Category of $\text{Tate}_\kappa^{el}(\mathcal{C})$.

Proposition 5.6. *The sub-category $\text{Pro}_\kappa^a(\mathcal{C}) \subset \text{Tate}_\kappa^{el}(\mathcal{C})$ is left s-filtering.*

Proof. The embedding $\text{Pro}_\kappa^a(\mathcal{C}) \hookrightarrow \text{Ind}_\kappa^a(\text{Pro}_\kappa^a(\mathcal{C}))$ is left s-filtering by Proposition 3.10. Elementary Tate objects form a full sub-category of $\text{Ind}_\kappa^a(\text{Pro}_\kappa^a(\mathcal{C}))$, so the result follows from Lemma 2.18. \square

5.2.2. *Exact Sequences of Elementary Tate Objects.* The proof of Proposition 3.12 implies the following.

Proposition 5.7. *The category $\text{Tate}_\kappa^{el}(\mathcal{EC})$ is canonically equivalent to $\mathcal{ETate}_\kappa^{el}(\mathcal{C})$.*

5.2.3. *Elementary Tate Objects as a Localization.* The proof of Proposition 3.14 implies the following.

Proposition 5.8 (See also [4], [21]). *Denote by $W \subset \mathbb{T}_\kappa(\mathcal{C})$ the sub-category consisting of all morphisms of elementary Tate diagrams given by strictly commuting triangles*

$$\begin{array}{ccc} I & \xrightarrow{\varphi} & J \\ & \searrow X & \swarrow Y \\ & \mathcal{C} & \end{array}$$

in which the map φ is final. The functor $\widehat{(-)}: \mathbb{T}_\kappa(\mathcal{C}) \longrightarrow \text{Tate}_\kappa^{el}(\mathcal{C})$ takes morphisms in W to isomorphisms of elementary Tate objects. The induced functor

$$\mathbb{T}_\kappa(\mathcal{C})[W^{-1}] \longrightarrow \text{Tate}_\kappa^{el}(\mathcal{C})$$

is an equivalence of categories.

We present a slight modification of this for later use.

Proposition 5.9. *Denote by $\mathbb{T}'_\kappa(\mathcal{C}) \subset \mathbb{T}_\kappa(\mathcal{C})$ the full sub-category of based elementary Tate diagrams, i.e. elementary Tate diagrams $X: I \longrightarrow \text{Pro}_\kappa^a(\mathcal{C})$ for which I has an initial object.¹¹ Define $W' \subset \mathbb{T}'_\kappa(\mathcal{C})$ to be the sub-category of final maps (i.e. $W' := W \cap \mathbb{T}'_\kappa(\mathcal{C})$).*

The restriction of $\widehat{(-)}$ to $\mathbb{T}'_\kappa(\mathcal{C})$ induces an equivalence of categories

$$(5.3) \quad \mathbb{T}'_\kappa(\mathcal{C})[W'^{-1}] \xrightarrow{\cong} \text{Tate}_\kappa^{el}(\mathcal{C})$$

Proof. The proof follows from Proposition 3.14 with minor changes. Given an elementary Tate diagram $X: I \longrightarrow \text{Pro}_\kappa^a(\mathcal{C})$, and $i \in I$, consider the sub-poset $I_i \subset I$ of all $j \geq i$ in I . The inclusion $I_i \hookrightarrow I$ is final. As a result, the diagram $X: I_i \longrightarrow \text{Pro}_\kappa^a(\mathcal{C})$ in $\mathbb{T}'_\kappa(\mathcal{C})$ also represents \widehat{X} , and (5.3) is essentially surjective.

Fullness follows by a slight modification of the straightening construction (Lemma 3.9). Let $X: I \longrightarrow \text{Pro}_\kappa^a(\mathcal{C})$ and $Y: J \longrightarrow \text{Pro}_\kappa^a(\mathcal{C})$ be based elementary Tate diagrams. Lemma 3.9 shows that any morphism $f: \widehat{X} \longrightarrow \widehat{Y}$ in $\text{Tate}_\kappa^{el}(\mathcal{C})$ is the colimit

¹¹We emphasize that we do not require that maps of based Tate diagrams map initial objects to initial objects.

of a span

$$\begin{array}{ccccc}
 I & \longleftarrow & I \downarrow_{\text{Pro}_\kappa^a(\mathcal{C})} J & \longrightarrow & J \\
 & \searrow X & \downarrow & \Rightarrow & \swarrow Y \\
 & & \text{Pro}_\kappa^a(\mathcal{C}) & &
 \end{array}$$

where the maps $I \downarrow_{\text{Pro}_\kappa^a(\mathcal{C})} J \longrightarrow I$ and $I \downarrow_{\text{Pro}_\kappa^a(\mathcal{C})} J \longrightarrow J$ are final. For any $(i, j, \gamma_{ij}) \in I \downarrow_{\text{Pro}_\kappa^a(\mathcal{C})} J$, the sub-poset $(I \downarrow_{\text{Pro}_\kappa^a(\mathcal{C})} J)_{(i, j, \gamma_{ij})} \subset I \downarrow_{\text{Pro}_\kappa^a(\mathcal{C})} J$ of all elements $(l, k, \gamma_{lk}) \geq (i, j, \gamma_{ij})$ is directed, final, and has an initial object. We see that the map f is the image of the morphism

$$\begin{array}{ccccc}
 I & \longleftarrow & (I \downarrow_{\text{Pro}_\kappa^a(\mathcal{C})} J)_{(i, j, \gamma_{ij})} & \longrightarrow & J \\
 & \searrow X & \downarrow & \Rightarrow & \swarrow Y \\
 & & \text{Pro}_\kappa^a(\mathcal{C}) & &
 \end{array}$$

in $\mathbb{T}'_\kappa(\mathcal{C})[W^{-1}]$.

Faithfulness follows by a slight modification of the argument for Proposition 3.14. Suppose that $X: I \longrightarrow \text{Pro}_\kappa^a(\mathcal{C})$ and $Y: J \longrightarrow \text{Pro}_\kappa^a(\mathcal{C})$ are based elementary Tate diagrams for which there exist morphisms

$$X \xrightleftharpoons[(\varphi_1, \alpha_1)]{(\varphi_0, \alpha_0)} Y$$

which induce equal maps of elementary Tate objects. For $a = 0, 1$, the pair (φ_a, α_a) induces a section of the map $I \downarrow_{\text{Pro}_\kappa^a(\mathcal{C})} J \longrightarrow I$. Denote by $i_0 \in I$ the initial object, and denote by $K_a \subset I \downarrow_{\text{Pro}_\kappa^a(\mathcal{C})} J$ the final sub-category consisting of all $(i, j, \gamma_{ij}) \geq (i_0, \varphi_a(i_0), \alpha_{a, i_0})$.

These sections fit into commuting triangles

$$\begin{array}{ccc}
 I & \xrightleftharpoons{(\varphi_a, \alpha_a)} & K_a \\
 & \searrow X & \swarrow \\
 & & \text{Pro}_\kappa^a(\mathcal{C})
 \end{array}$$

The existence of these commuting triangles implies that, for $a = 0, 1$, the image of the map $(\varphi_a, \alpha_a): X \longrightarrow Y$ in the localization $\mathbb{T}'_\kappa(\mathcal{C})[W^{-1}]$ is equal to the map represented by the zig-zag

$$\begin{array}{ccccc}
 I & \longleftarrow & K_a & \longrightarrow & J \\
 & \searrow X & \downarrow & \Rightarrow & \swarrow Y \\
 & & \text{Pro}_\kappa^a(\mathcal{C}) & &
 \end{array}$$

Let $b \in I \downarrow_{\text{Pro}_\kappa^a(\mathcal{C})} J$ be an element with $b \geq (i_0, \varphi_a(i_0), \alpha_{a, i_0})$ for $a = 0, 1$. Denote by $K \subset I \downarrow_{\text{Pro}_\kappa^a(\mathcal{C})} J$ the final sub-category on all $(i, j, \gamma_{ij}) \geq b$. For $a = 0, 1$, we

have $K \subset K_a$ with K final. This implies that the maps represented by the zig-zags above are isomorphic to the zig-zag

$$\begin{array}{ccccc} I & \xleftarrow{\quad} & K & \xrightarrow{\quad} & J \\ & \searrow X & \downarrow & \Rightarrow & \swarrow Y \\ & & \text{Pro}_\kappa^a(\mathcal{C}) & & \end{array}$$

We conclude that (5.3) is faithful. \square

5.2.4. Functoriality of the Construction.

Proposition 5.10. *An exact functor $F: \mathcal{C} \rightarrow \mathcal{D}$ extends canonically to an exact functor*

$$\text{Tate}_\kappa^{el}(\mathcal{C}) \xrightarrow{\tilde{F}} \text{Tate}_\kappa^{el}(\mathcal{D}).$$

If F is faithful or fully faithful, then so is \tilde{F} .

Proof. Proposition 3.15 and the analogous clause in Theorem 4.2 show that F extends canonically to an exact functor $\tilde{F}: \text{Ind}_\kappa^a(\text{Pro}_\kappa^a(\mathcal{C})) \rightarrow \text{Ind}_\kappa^a(\text{Pro}_\kappa^a(\mathcal{D}))$ such that \tilde{F} is (fully) faithful if F is. It suffices to show that \tilde{F} preserves elementary Tate objects.

Represent $\hat{X} \in \text{Tate}_\kappa^{el}(\mathcal{C})$ by an elementary Tate diagram of size at most κ

$$I \xrightarrow{X} \text{Pro}_\kappa^a(\mathcal{C}).$$

Because F is exact, the diagram

$$I \xrightarrow{FX} \text{Pro}_\kappa^a(\mathcal{D})$$

is also an elementary Tate diagram. We conclude that $\tilde{F}(\hat{X}) \in \text{Tate}_\kappa^{el}(\mathcal{D})$. \square

5.2.5. Countable Elementary Tate Objects. Countable elementary Tate objects were introduced by Beilinson [4, A.3]. We show here that our approach is compatible with his.

Definition 5.11. (Beilinson [4, A.3]) Let \mathcal{C} be an exact category. Let $\Pi \subset \mathbb{Z} \times \mathbb{Z}$ be the full sub-poset consisting of all (i, j) with $i \leq j$. An *admissible Π -diagram* in \mathcal{C} is a functor $X: \Pi \rightarrow \mathcal{C}$ such that for all $i \leq j \leq k$, the sequence

$$X_{i,j} \hookrightarrow X_{i,k} \twoheadrightarrow X_{j,k}$$

is short exact. Denote by $\Pi^a(\mathcal{C})$ the category of admissible Π -diagrams in \mathcal{C} and natural transformations between them.

Definition 5.12. A functor $\varphi: \mathbb{Z} \rightarrow \mathbb{Z}$ is *bifinal* if $\varphi(n) \rightarrow \pm\infty$ as $n \rightarrow \pm\infty$. Let φ_0 and φ_1 be two bifinal maps. We say $\varphi_0 \leq \varphi_1$ if, for all $n \in \mathbb{Z}$, $\varphi_0(n) \leq \varphi_1(n)$.

A functor $\varphi: \mathbb{Z} \rightarrow \mathbb{Z}$ induces a functor $\varphi: \Pi \rightarrow \Pi$ by applying φ in each factor.

Definition 5.13. Denote by $U \subset \Pi^a(\mathcal{C})$ the sub-category consisting of all morphisms of the form $X\varphi_0 \rightarrow X\varphi_1$ for bifinal maps $\varphi_0 \leq \varphi_1$. The *Beilinson category* $\lim_{\leftrightarrow} \mathcal{C}$ is the localization $\Pi^a(\mathcal{C})[U^{-1}]$.

Proposition 5.14. (*Previdi [21, Theorem 5.8]*) *Let \mathcal{C} be an exact category. The Beilinson category $\lim_{\leftrightarrow} \mathcal{C}$ embeds as a full sub-category of $\text{Ind}_{\mathbb{N}_0}^a(\text{Pro}_{\mathbb{N}_0}^a(\mathcal{C}))$.*

Remark 5.15. Let $X: \Pi \longrightarrow \mathcal{C}$ be an admissible Π -diagram in \mathcal{C} . The assignment

$$\{X_{i,j}\} \mapsto \text{colim}_j \lim_i X_{i,j}$$

extends to a functor $\lim_{\leftrightarrow} \mathcal{C} \longrightarrow \text{Ind}_{\mathbb{N}_0}^a(\text{Pro}_{\mathbb{N}_0}^a(\mathcal{C}))$. Previdi [21] shows that this is a fully faithful embedding into the countable envelope of the dual of the countable envelope of \mathcal{C}^{op} (Previdi denotes this by $\text{IP}^a(\mathcal{C})$). Proposition 3.17 and its analogue for countable admissible Pro-objects imply that $\text{IP}^a(\mathcal{C})$ is equivalent to the category $\text{Ind}_{\mathbb{N}_0}^a(\text{Pro}_{\mathbb{N}_0}^a(\mathcal{C}))$.

Proposition 5.16. *Let \mathcal{C} be an exact category. The Beilinson category $\lim_{\leftrightarrow} \mathcal{C}$ is equivalent to $\text{Tate}_{\mathbb{N}_0}^{el}(\mathcal{C})$.*

Proof. We show that the embedding $\lim_{\leftrightarrow} \mathcal{C} \hookrightarrow \text{Ind}_{\mathbb{N}_0}^a(\text{Pro}_{\mathbb{N}_0}^a(\mathcal{C}))$ takes its essential image in the category $\text{Tate}_{\mathbb{N}_0}^{el}(\mathcal{C})$. Let $X: \Pi \longrightarrow \mathcal{C}$ be an admissible Π -diagram in \mathcal{C} representing $\widehat{X} \in \lim_{\leftrightarrow}(\mathcal{C}) \subset \text{Ind}_{\mathbb{N}_0}^a(\text{Pro}_{\mathbb{N}_0}^a(\mathcal{C}))$.

The definition of admissible Π -diagram ensures that the assignment

$$n \mapsto X_{-n,0}$$

defines a countable admissible Pro-diagram $X_{*,0}: \mathbb{N} \longrightarrow \mathcal{C}^{op}$. Denote by \widehat{X}_0 the associated admissible Pro-object. The canonical map $\widehat{X}_0 \hookrightarrow \widehat{X}$ is an admissible monic (Lemma 3.11). The quotient $\widehat{X}/\widehat{X}_0$ is an admissible Ind-object. Indeed, the quotient is represented by the admissible Π -diagram $X/X_0: \Pi \longrightarrow \mathcal{C}$ which, for $j < 0$, sends (i, j) to $0 \in \mathcal{C}$ and, for $j \geq 0$, sends (i, j) to $X_{i,j}/X_{i,0}$. For any $i \leq 0 \leq j$, we have that $X_{i,j}/X_{i,0} \cong X_{0,j}$ because X is an admissible Π -diagram. In particular, we see that X/X_0 is constant in the Pro-direction (the first factor of Π). We conclude that $\widehat{X}/\widehat{X}_0 \in \text{Ind}_{\mathbb{N}_0}^a(\mathcal{C})$. Using Theorem 5.4, we conclude that $\widehat{X} \in \text{Tate}_{\mathbb{N}_0}^{el}(\mathcal{C})$.

Conversely, every countable elementary Tate object is an extension of a countable admissible Ind-object by a countable admissible Pro-object. Proposition 3.17 shows that the categories $\text{Ind}_{\mathbb{N}_0}^a(\mathcal{C})$ and $\text{Pro}_{\mathbb{N}_0}^a(\mathcal{C})$ are contained in the Beilinson category. The proof of Previdi's [21, Theorem 6.1] shows that the Beilinson category is closed under extensions in the category of countable admissible Ind-Pro objects. We conclude that every elementary Tate object is isomorphic to an object in $\lim_{\leftrightarrow}(\mathcal{C})$. \square

For countable elementary Tate objects, we also have the following.

Proposition 5.17. *The category $\text{Tate}_{\mathbb{N}_0}^{el}(\mathcal{C})$ is split exact if \mathcal{C} is.*

Proof. Proposition 3.18 implies that $\text{Pro}_{\mathbb{N}_0}^a(\mathcal{C})$ and $\text{Ind}_{\mathbb{N}_0}^a(\text{Pro}_{\mathbb{N}_0}^a(\mathcal{C}))$ are split exact if \mathcal{C} is. As a fully exact sub-category of $\text{Ind}_{\mathbb{N}_0}^a(\text{Pro}_{\mathbb{N}_0}^a(\mathcal{C}))$, $\text{Tate}_{\mathbb{N}_0}^{el}(\mathcal{C})$ is split exact as well. \square

Proposition 5.18. *Let \mathcal{C} be a split exact category for which there exists a collection of objects $\{S_i\}_{i \in \mathbb{N}} \subset \mathcal{C}$ such that every object $Y \in \mathcal{C}$ is a direct summand of $\bigoplus_{i=0}^n S_i$*

for some n . Denote by $\widehat{\prod_{\mathbb{N}} S}$ and $\widehat{\bigoplus_{\mathbb{N}} S}$ the admissible Pro and Ind-objects

$$\begin{aligned}\widehat{\prod_{\mathbb{N}} S} &:= \lim_{(m,n) \in \mathbb{N} \times \mathbb{N}} \prod_{i=0}^m S_i^{\times n}, \text{ and} \\ \widehat{\bigoplus_{\mathbb{N}} S} &:= \operatorname{colim}_{(m,n) \in \mathbb{N} \times \mathbb{N}} \bigoplus_{i=0}^m S_i^{\oplus n}.\end{aligned}$$

Then every countable elementary Tate object in \mathcal{C} is a direct summand of

$$\widehat{\prod_{\mathbb{N}} S} \oplus \widehat{\bigoplus_{\mathbb{N}} S}.$$

Proof. The proof of Theorem 5.4 shows that every elementary Tate object \widehat{X} fits into an exact sequence

$$\widehat{L} \hookrightarrow \widehat{X} \twoheadrightarrow \widehat{X/L}$$

where $\widehat{L} \in \operatorname{Pro}_{\aleph_0}^a(\mathcal{C})$ and $\widehat{X/L} \in \operatorname{Ind}_{\aleph_0}^a(\mathcal{C})$. By Proposition 3.19, $\widehat{X/L}$ is a direct summand of $\widehat{\bigoplus_{\mathbb{N}} S}$. The analogous result for Pro objects (in Theorem 4.2) shows that \widehat{L} is a direct summand of $\widehat{\prod_{\mathbb{N}} S}$. \square

Example 5.19. Let R be a ring. The category $\operatorname{Tate}_{\aleph_0}^{el}(P^f(R))$ is split exact, and every countable elementary Tate module is a direct summand of $R((t))$.

5.3. The Category of Tate Objects.

Definition 5.20. Let \mathcal{C} be an idempotent complete exact category. Define the category $\operatorname{Tate}_{\kappa}(\mathcal{C})$ of Tate objects in \mathcal{C} of size at most κ to be the idempotent completion of $\operatorname{Tate}_{\kappa}^{el}(\mathcal{C})$.

The discussion of Section 3.2.6 shows that $\operatorname{Tate}_{\kappa}^{el}(\mathcal{C})$ does not coincide with $\operatorname{Tate}_{\kappa}(\mathcal{C})$ in general. Drinfeld [10, Example 3.2.2.2] provides another example of a Tate module over a commutative ring which is not elementary.

5.4. Tate R-Modules. The categories of admissible Ind, Pro and Tate objects in finite dimensional vector spaces over a discrete field date back at least to Lefschetz [16, Chapter II.25]. Lefschetz described these as categories of topological vector spaces, which he called “discrete”, “linearly compact” and “locally linearly compact” respectively. More recently, Drinfeld [10] has introduced a notion of Tate modules over discrete rings R . We relate his notion to Tate objects in the category $P^f(R)$ of finitely generated projective (left) R -modules.

Definition 5.21 (Drinfeld). Let R be a ring. An *elementary Tate module à la Drinfeld* is a topological R -module isomorphic to $P \oplus Q^{\vee}$ where P is a discrete projective (left) module and Q^{\vee} is the topological dual of a discrete projective (right) module. A *Tate module à la Drinfeld* is a topological direct summand of an elementary Tate module $P \oplus Q^{\vee}$.

Denote by $\operatorname{Tate}^{Dr}(R)$ the category of Tate modules à la Drinfeld and continuous homomorphisms. Denote by $\operatorname{Tate}_{\aleph_0}^{Dr}(R) \subset \operatorname{Tate}^{Dr}(R)$ the full sub-category on direct summands of modules $P \oplus Q^{\vee}$ where P and Q are countably generated.

Theorem 5.22. *The category $\mathbf{Tate}^{Dr}(R)$ of Tate modules à la Drinfeld is a full sub-category of $\mathbf{Tate}(P^f(R))$. The categories $\mathbf{Tate}_{\aleph_0}^{Dr}(R)$ and $\mathbf{Tate}_{\aleph_0}(P^f(R))$ are equivalent.*

Proof of Theorem 5.22. The category $\mathbf{Ind}^a(\mathbf{Pro}^a(P^f(R)))$ embeds fully faithfully in the category $\mathbf{Mod}(R)_{top}$ of topological R -modules, by an iterate of the construction in the proof of Proposition 4.4. This extends to a fully faithful embedding $(\mathbf{Ind}^a(\mathbf{Pro}^a(P^f(R))))^{ic} \hookrightarrow \mathbf{Mod}(R)_{top}$, because $\mathbf{Mod}(R)_{top}$ is idempotent complete. In particular, $\mathbf{Tate}(P^f(R))$ embeds as a full sub-category of $\mathbf{Mod}(R)_{top}$. We show that this sub-category contains $\mathbf{Tate}^{Dr}(R)$.

It suffices to show that every elementary Tate module à la Drinfeld $P \oplus Q^\vee$ is in $\mathbf{Tate}^{el}(P^f(R))$. The module $P \oplus Q^\vee$ is trivially an extension of the discrete projective (left) module P by the topological dual of a discrete projective (right) module Q . Proposition 3.24 and Corollary 4.6 show that P and Q^\vee are objects in $\mathbf{Tate}(P^f(R))$, so $P \oplus Q^\vee$ is as well.

To prove $\mathbf{Tate}_{\aleph_0}^{Dr}(R) \simeq \mathbf{Tate}_{\aleph_0}(P^f(R))$, we show that every countable elementary Tate object in $P^f(R)$ is elementary à la Drinfeld. Indeed, every countable elementary Tate module $V \in \mathbf{Tate}_{\aleph_0}^{el}(P^f(R))$ admits a lattice L . The lattice L is isomorphic to the topological dual of a discrete, countably generated, projective (right) module by Corollary 4.6. The quotient V/L is discrete and projective by Proposition 3.24. \square

Remark 5.23. It is possible to give a purely categorical description of $\mathbf{Tate}^{Dr}(R)$ along the lines of Corollary 3.29. For idempotent complete \mathcal{C} , denote by

$$\widetilde{\mathbf{Ind}^a(\mathcal{C})} \subset \mathbf{Ind}^a(\mathcal{C})$$

the full sub-category consisting of arbitrary direct sums of objects in $\mathbf{Ind}_{\aleph_0}^a(\mathcal{C})$. Denote by

$$\widetilde{\mathbf{Pro}^a(\mathcal{C})} \subset \mathbf{Pro}^a(\mathcal{C})$$

the full sub-category consisting of arbitrary direct products of objects in $\mathbf{Pro}_{\aleph_0}^a(\mathcal{C})$. Define

$$\mathbf{Tate}^{Dr}(\mathcal{C}) \subset \mathbf{Tate}(\mathcal{C})$$

to be the idempotent completion of the smallest full sub-category of $\mathbf{Tate}^{el}(\mathcal{C})$ which contains the categories $\widetilde{\mathbf{Pro}^a(\mathcal{C})}$ and $\widetilde{\mathbf{Ind}^a(\mathcal{C})}$ and which is closed under extensions. Our discussion above shows that for $\mathcal{C} = P^f(R)$, $\mathbf{Tate}^{Dr}(\mathcal{C}) \simeq \mathbf{Tate}^{Dr}(R)$.

5.5. Tate Objects and the Kalkin Category. Let R be a ring. Denote by $\pi_{R[[t]]}: R((t)) \longrightarrow R[[t]]$ the projection onto $R[[t]] \subset R((t))$. Tate objects provide a categorical analogue of the R -algebra

$$\mathbf{Mat}_{\infty}^{\pm}(R) := \{A \in \mathbf{End}_R(R((t))) \mid [A, \pi_{R[[t]]}] \text{ has finite rank.}\}$$

The assignment $A \mapsto (1 - \pi_{R[[t]]})A(1 - \pi_{R[[t]]})$ defines a surjective algebra homomorphism

$$\mathbf{Mat}_{\infty}^{\pm}(R) \longrightarrow \mathbf{Calk}(R).$$

This homomorphism admits a categorical analogue.

5.5.1. *The Map* $\mathbf{Tate}_\kappa(\mathcal{C}) \rightarrow \mathbf{Calk}_\kappa(\mathcal{C})$. Recall from Proposition 5.9 that $\mathbb{T}'_\kappa(\mathcal{C})$ is the category of based elementary Tate diagrams, $X : I \longrightarrow \mathbf{Pro}^a(\mathcal{C})$. Denote by $i_0 \in I$ the initial object. The assignment

$$\begin{array}{ccc} I & & I \\ \downarrow X & \longmapsto & \downarrow X/X_{i_0} \\ \mathbf{Pro}_\kappa^a(\mathcal{C}) & & \mathcal{C} \end{array}$$

extends to a functor

$$\mathbb{T}'_\kappa(\mathcal{C}) \xrightarrow{q} \mathbf{Calk}_\kappa(\mathcal{C})$$

This functor sends a morphism

$$\begin{array}{ccc} I & \xrightarrow{\psi} & J \\ & \searrow X \quad \alpha \quad \swarrow Y & \\ & \mathbf{Pro}_\kappa^a(\mathcal{C}) & \end{array}$$

to the morphism

$$\mathrm{colim}_I X_i/X_{i_0} \longrightarrow \mathrm{colim}_I Y_{\psi(i)}/Y_{\psi(i_0)} \xrightarrow{\cong} \mathrm{colim}_J Y_j/Y_{j_0}$$

The isomorphism on the right is the inverse, in $\mathbf{Calk}_\kappa(\mathcal{C})$, of the map in $\mathbf{Calk}_\kappa(\mathcal{C})$ given by the zig-zag

$$\mathrm{colim}_I Y_{\psi(i)}/Y_{\psi(i_0)} \longleftarrow \mathrm{colim}_I Y_{\psi(i)}/Y_{j_0} \longrightarrow \mathrm{colim}_J Y_j/Y_{j_0}$$

Recall that $W' \subset \mathbb{T}'_\kappa(\mathcal{C})$ denotes the sub-category of final maps (which are not required to preserve the initial object!). By inspection, q takes maps in W' to isomorphisms. Proposition 5.9 and the universal property of localization guarantee that q induces a unique functor

$$(5.4) \quad \mathbf{Tate}_\kappa^{el}(\mathcal{C}) \longrightarrow \mathbf{Calk}_\kappa(\mathcal{C}) .$$

Our construction guarantees that (5.4) is exact. By the universal property of idempotent completion, (5.4) extends uniquely to an exact functor

$$\mathbf{Tate}_\kappa(\mathcal{C}) \xrightarrow{\tilde{q}} \mathbf{Calk}_\kappa(\mathcal{C}) .$$

5.5.2. *The Kernel of* $\mathbf{Tate}_\kappa(\mathcal{C}) \rightarrow \mathbf{Calk}_\kappa(\mathcal{C})$. If $A \in \mathbf{Mat}_\infty^\pm(R)$ has its image contained in $R[[t]] \oplus M \subset R((t))$ for some finitely generated sub-module M , then $A \in \ker(\mathbf{Mat}_\infty^\pm(R) \longrightarrow \mathbf{Calk}(R))$. Similarly, the construction of \tilde{q} shows that it takes admissible Pro-objects to the zero object in $\mathbf{Calk}_\kappa(\mathcal{C})$. As a result, it takes admissible epics (monics) in $\mathbf{Tate}_\kappa(\mathcal{C})$ whose (co)kernels are in $\mathbf{Pro}_\kappa^a(\mathcal{C})^{ic}$ to isomorphisms. It therefore factors through an exact functor

$$(\mathbf{Tate}_\kappa(\mathcal{C})/\mathbf{Pro}_\kappa^a(\mathcal{C})^{ic})^{ic} \xrightarrow{\tilde{q}} \mathbf{Calk}_\kappa(\mathcal{C}) .$$

Proposition 5.24. (See also Saito [24, Lemma 3.5]) *The map*

$$(\mathrm{Tate}_\kappa(\mathcal{C})/\mathrm{Pro}_\kappa^a(\mathcal{C})^{ic})^{ic} \xrightarrow{\tilde{q}} \mathrm{Calk}_\kappa(\mathcal{C})$$

is an equivalence of exact categories.

Proof. The universal property of localization and idempotent completion ensure that the map $\mathrm{Ind}_\kappa^a(\mathcal{C}) \hookrightarrow \mathrm{Tate}_\kappa(\mathcal{C}) \longrightarrow (\mathrm{Tate}_\kappa(\mathcal{C})/\mathrm{Pro}_\kappa^a(\mathcal{C})^{ic})^{ic}$ induces a canonical map

$$\mathrm{Calk}_\kappa(\mathcal{C}) \xrightarrow{\iota} (\mathrm{Tate}_\kappa(\mathcal{C})/\mathrm{Pro}_\kappa^a(\mathcal{C})^{ic})^{ic}$$

We will show that ι is inverse to \tilde{q} . From the construction of idempotent completion, it suffices to show that:

- (1) if $\hat{Y} \in \mathrm{Calk}_\kappa(\mathcal{C})$ is isomorphic to an admissible Ind-object, then $\tilde{q}\iota(\hat{Y})$ is naturally isomorphic to \hat{Y} , and
- (2) if $\hat{X} \in (\mathrm{Tate}_\kappa(\mathcal{C})/\mathrm{Pro}_\kappa^a(\mathcal{C})^{ic})^{ic}$ is isomorphic to an elementary Tate object, then $\iota\tilde{q}(\hat{X})$ is naturally isomorphic to \hat{X} .

The first is immediate from the construction of ι and \tilde{q} . Conversely, if \hat{X} is an elementary Tate object, the construction of \tilde{q} defines a map

$$\hat{Y} \longrightarrow \iota\tilde{q}(\hat{Y})$$

whose kernel is an admissible Pro-object. □

Similar arguments show the following.

Proposition 5.25. *The assignment $X \mapsto X_{i_0}$ extends to a functor*

$$\mathbb{T}'_\kappa(\mathcal{C}) \longrightarrow \mathrm{Pro}_\kappa^a(\mathcal{C})/\mathcal{C}$$

This induces an equivalence of exact categories

$$\mathrm{Tate}_\kappa(\mathcal{C})/\mathrm{Ind}_\kappa^a(\mathcal{C}) \xrightarrow{\cong} \mathrm{Pro}_\kappa^a(\mathcal{C})/\mathcal{C}$$

Its inverse $\mathrm{Pro}_\kappa^a(\mathcal{C})/\mathcal{C} \hookrightarrow \mathrm{Tate}_\kappa^{el}(\mathcal{C})/\mathrm{Ind}_\kappa^a(\mathcal{C})$ is the map induced by the inclusion $\mathrm{Pro}_\kappa^a(\mathcal{C}) \subset \mathrm{Tate}_\kappa^{el}(\mathcal{C})$.

Remark 5.26. As S. Saito [24] has observed in the countable case, the propositions above combine with the Eilenberg swindle and Schlichting's localization theorem [27] to show that $K_i(\mathrm{Tate}_\kappa(\mathcal{C})) \cong K_{i-1}(\mathcal{C})$, when \mathcal{C} is idempotent complete.

6. SATO GRASSMANNIANS

Let k be a field and consider the Tate vector space $k((t))$. Sato and Sato [26] introduced an infinite dimensional Grassmannian $Gr(k((t)))$ whose points correspond to *lattices*, i.e. members of a certain class of subspaces $L \subset k((t))$. They then constructed a *determinant line bundle* $\mathcal{L} \longrightarrow Gr(k((t)))$ and employed this to great effect in applications.

In this section, we recall the definition of lattices and Sato Grassmannians for elementary Tate objects in general exact categories (see also Previdi [22]). The key properties required for the construction of the determinant line are:

- (1) for any nested pair of lattices $L_0 \subset L_1 \subset k((t))$, the quotient L_1/L_0 is finite dimensional, and

- (2) for any pair of lattices L_0 and L_1 , there exists a common enveloping lattice N with $L_i \subset N$ for $i = 0, 1$.

We show that, for any \mathcal{C} , the construction of the Grassmannian is natural with respect to exact functors, and that the analogue of the first property holds. If \mathcal{C} is idempotent complete, we show that the analogue of the second holds as well.

Definition 6.1. Let $V \in \mathbf{Tate}_\kappa^{el}(\mathcal{C})$. A *lattice* L in V consists of an admissible monic

$$L \hookrightarrow V$$

such that $L \in \mathbf{Pro}_\kappa^a(\mathcal{C})$ and $V/L \in \mathbf{Ind}_\kappa^a(\mathcal{C})$. The *Sato Grassmannian* $Gr(V)$ is the poset of lattices of V ordered by inclusion.

Remark 6.2.

- (1) We use the term “lattice” to refer to what Drinfeld [10] calls a “coprojective lattice.”
- (2) The proof of Theorem 5.4 shows that every elementary Tate object has a lattice, and further that any object in an elementary Tate diagram is a lattice of the associated Tate object.

Proposition 6.3. Let V be an elementary Tate object in \mathcal{C} . An exact functor $F: \mathcal{C} \rightarrow \mathcal{D}$ induces an order preserving map

$$\begin{aligned} Gr(V) &\xrightarrow{F_V} Gr(\tilde{F}(V)) . \\ L &\longmapsto \tilde{F}(L) \end{aligned}$$

Proof. Order preserving is clear once we show that F_V is well defined. Let

$$L \hookrightarrow V$$

be a lattice in V . It defines an exact sequence

$$L \hookrightarrow V \twoheadrightarrow V/L$$

of elementary Tate objects in \mathcal{C} , where V/L is an admissible Ind-object. The extension of F to elementary Tate objects preserves exact sequences as well as admissible Pro and Ind-objects, so

$$\tilde{F}(L) \hookrightarrow \tilde{F}(V) \twoheadrightarrow \tilde{F}(V/L)$$

is an exact sequence of elementary Tate objects in \mathcal{D} , such that $\tilde{F}(L)$ is an admissible Pro-object, and such that $\tilde{F}(V/L)$ is an admissible Ind-object. \square

Remark 6.4. Let R be a commutative ring. Let V be an elementary Tate R -module. Proposition 6.3 shows that the Sato Grassmannian $Gr(V)$ defines a presheaf over $\mathrm{Spec}(R)$. The Sato Grassmannian $Gr(V)$ can in fact be viewed as an Ind-algebraic space which is Ind-proper over $\mathrm{Spec}(R)$ [10, Proposition 3.8].

Proposition 6.5. Let $L_0 \hookrightarrow L_1 \hookrightarrow V$ be a nested pair of lattices in $V \in \mathbf{Tate}_\kappa^{el}(\mathcal{C})$. The quotient L_1/L_0 is an object of \mathcal{C} .

Proof. We show that L_1/L_0 lies in $\mathbf{Ind}_\kappa^a(\mathcal{C}) \cap \mathbf{Pro}_\kappa^a(\mathcal{C}) \subset \mathbf{Tate}_\kappa^{el}(\mathcal{C})$, and that this intersection is the sub-category $\mathcal{C} \subset \mathbf{Tate}_\kappa^{el}(\mathcal{C})$.

By Noether's Lemma [9, Lemma 3.5], a nested pair of lattices gives rise to an exact sequence

$$L_1/L_0 \hookrightarrow V/L_0 \twoheadrightarrow V/L_1.$$

The object L_1/L_0 is an admissible Pro-object. Indeed, it is an admissible quotient in $\mathbf{Tate}_\kappa^{el}(\mathcal{C})$ of an admissible Pro-object, and $\mathbf{Pro}_\kappa^a(\mathcal{C}) \subset \mathbf{Tate}_\kappa^{el}(\mathcal{C})$ is left s-filtering. The object L_1/L_0 is also an admissible Ind-object. Indeed, by straightening exact sequences of admissible Ind-Pro objects, we see that, in an exact sequence of admissible Ind-Pro objects, the central term is an admissible Ind-object if and only if the outer two terms are.

Let $L: I \longrightarrow \mathcal{C}$ be an admissible Ind-diagram representing L_1/L_0 . Because L_1/L_0 is also an admissible Pro-object, the isomorphism $L_1/L_0 \cong \operatorname{colim}_I L_i$ factors through the inclusion of L_i for some i

$$\begin{array}{ccc} L_1/L_0 & \xrightarrow{\cong} & \operatorname{colim}_I L_i \\ & \searrow & \uparrow \\ & & L_i \end{array}$$

The inclusion is therefore an epic admissible monic, i.e. an isomorphism. We conclude that L_1/L_0 is in \mathcal{C} . \square

Theorem 6.6. *Let \mathcal{C} be idempotent complete. Let $V \in \mathbf{Tate}_\kappa^{el}(\mathcal{C})$. The Sato Grassmannian $\operatorname{Gr}(V)$ is a directed poset.*

Proof. Let L_0 and L_1 be two lattices of V . We show that there exists a lattice N of V and admissible monics $L_i \hookrightarrow N$ over V for $i = 0, 1$. Let $V: I \longrightarrow \mathbf{Pro}_\kappa^a(\mathcal{C})$ be an elementary Tate diagram representing V . Because $L_0 \oplus L_1$ is an admissible Pro-object, there exists $i \in I$, such that the morphism $L_0 \oplus L_1 \longrightarrow V$ factors through the admissible monic $V_i \hookrightarrow V$. Note that $V_i \hookrightarrow V$ is the inclusion of a lattice. Define $N := V_i \hookrightarrow V$. When \mathcal{C} is idempotent complete, we show that the maps $L_i \longrightarrow N$ are admissible monics in $\mathbf{Pro}_\kappa^a(\mathcal{C})$.

In an idempotent complete exact category, a map is an admissible monic if its post-composition by an admissible monic is an admissible monic (by the dual to [9, Corollary 7.7]). This implies that the maps $L_i \longrightarrow N$ are admissible monics in $\mathbf{Tate}_\kappa(\mathcal{C})$.

The category of $\mathbf{Pro}_\kappa^a(\mathcal{C})$ is closed under extensions in $\mathbf{Tate}_\kappa(\mathcal{C})$ (it is left s-filtering). To conclude that the maps $L_i \longrightarrow N$ are admissible monics, it is therefore enough to show that the cokernels N/L_i are objects in $\mathcal{C} \subset \mathbf{Tate}_\kappa(\mathcal{C})$.

An analogue of the proof of Proposition 6.5 implies that N/L_i is contained in $\mathbf{Pro}_\kappa^a(\mathcal{C})^{ic} \cap \mathbf{Ind}_\kappa^a(\mathcal{C})^{ic} \subset \mathbf{Lex}(\mathbf{Pro}_\kappa^a(\mathcal{C}))$. We show that, when \mathcal{C} is idempotent complete, this intersection is the sub-category $\mathcal{C} \subset \mathbf{Lex}(\mathbf{Pro}_\kappa^a(\mathcal{C}))$.

By assumption, there exist objects $P, Q \in \mathbf{Lex}(\mathbf{Pro}_\kappa^a(\mathcal{C}))$ such that $N/L_0 \oplus P \in \mathbf{Pro}_\kappa^a(\mathcal{C})$ and $N/L_0 \oplus Q \in \mathbf{Ind}_\kappa^a(\mathcal{C})$.

The composition

$$(6.1) \quad N/L_0 \oplus P \twoheadrightarrow N/L_0 \hookrightarrow N/L_0 \oplus Q$$

is a morphism of admissible Ind-Pro objects, because the embedding of an exact category into its idempotent completion is fully faithful. Let $X: I \longrightarrow \mathcal{C}$ be an

admissible Ind-diagram representing $N/L_0 \oplus Q$. Because $N/L_0 \oplus P \in \text{Pro}_\kappa^a(\mathcal{C})$, there exists $i \in I$ for which the map 6.1 factors as

$$\begin{array}{ccccc} N/L_0 \oplus P & \longrightarrow & N/L_0 & \hookrightarrow & N/L_0 \oplus Q \\ & \searrow f & \downarrow \exists \tilde{f} & & \nearrow \\ & & X_i & \hookrightarrow & \end{array} .$$

Note that the map \tilde{f} is induced from the map f by the universal property of cokernels. The object N/L_0 is a retract of X_i , in $\text{Lex}(\text{Pro}_\kappa^a(\mathcal{C}))$, because N/L_0 is a retract of $L_2/L_0 \oplus Q$. The composite

$$X_i \longrightarrow N/L_0 \longrightarrow X_i$$

is an idempotent. If \mathcal{C} is idempotent complete, this idempotent splits, and we conclude that N/L_0 is an object of \mathcal{C} . The same reasoning applies to N/L_1 . \square

7. N-TATE OBJECTS

In this section we consider n -Tate objects, and record their basic properties. We then recall the Beilinson–Parshin theory of n -dimensional adèles and we show that the n -dimensional adèles of an n -dimensional scheme are naturally an n -Tate object.

7.1. The Category of n -Tate Objects and its Properties.

Definition 7.1. Let \mathcal{C} be idempotent complete. Define the category $\text{n-Tate}_\kappa^{el}(\mathcal{C})$ of elementary n -Tate objects of size at most κ by

$$\text{n-Tate}_\kappa^{el}(\mathcal{C}) := \text{Tate}_\kappa^{el}((n-1)\text{-Tate}_\kappa(\mathcal{C}))$$

The category $\text{n-Tate}_\kappa(\mathcal{C})$ of n -Tate objects of size at most κ is the idempotent completion of $\text{n-Tate}_\kappa^{el}(\mathcal{C})$.

Example 7.2. Define a 0-dimensional local field to be a finite field. Let k be a finite field. Define an n -dimensional local field over k to be a complete discrete valuation field F with ring of integers R such that the residue field of R is an $(n-1)$ -dimensional local field over k . Vector spaces over n -dimensional local fields are canonically elementary n -Tate objects in the category of finitely generated abelian groups.

The results of Sections 5 and 6 carry over to n -Tate objects.

Theorem 7.3. Let \mathcal{C} be an idempotent complete exact category.

- (1) $\text{Tate}_\kappa^{el}(\mathcal{C})$ is the smallest exact sub-category of $\text{Ind}_\kappa^a(\text{Pro}_\kappa^a((n-1)\text{-Tate}_\kappa(\mathcal{C})))$ which contains $\text{Pro}_\kappa^a((n-1)\text{-Tate}_\kappa(\mathcal{C}))$, and $\text{Ind}_\kappa^a((n-1)\text{-Tate}_\kappa(\mathcal{C}))$ and which is closed under extensions.
- (2) The categories $\text{n-Tate}_\kappa^{el}(\mathcal{EC})$ and $\mathcal{E}(\text{n-Tate}_\kappa^{el}(\mathcal{C}))$ are canonically equivalent. The categories $\text{n-Tate}_\kappa(\mathcal{EC})$ and $\mathcal{E}(\text{n-Tate}_\kappa(\mathcal{C}))$ are canonically equivalent as well.

(3) An exact functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ extends canonically to a pair of exact functors

$$\begin{array}{ccc} \mathbf{n}\text{-Tate}_{\kappa}^{el}(\mathcal{C}) & \xrightarrow{\tilde{F}} & \mathbf{n}\text{-Tate}_{\kappa}^{el}(\mathcal{D}) \\ \downarrow & & \downarrow \\ \mathbf{n}\text{-Tate}_{\kappa}(\mathcal{C}) & \xrightarrow{\tilde{F}} & \mathbf{n}\text{-Tate}_{\kappa}(\mathcal{D}) \end{array}$$

If F is faithful or fully faithful, then so are both functors \tilde{F} .

- (4) The sub-category $\text{Pro}_{\kappa}^a((\mathbf{n}-1)\text{-Tate}_{\kappa}(\mathcal{C})) \subset \mathbf{n}\text{-Tate}_{\kappa}^{el}(\mathcal{C})$ is left s -filtering.
- (5) The category $\mathbf{n}\text{-Tate}_{\mathbb{N}_0}^{el}(\mathcal{C})$ is split exact if \mathcal{C} is.
- (6) Every elementary n -Tate object has a lattice.
- (7) Let V be an elementary n -Tate object in \mathcal{C} . An exact functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ induces an order-preserving map

$$\begin{array}{ccc} \text{Gr}(V) & \xrightarrow{F_V} & \text{Gr}(\tilde{F}(V)) \\ L \longmapsto & & \tilde{F}(L) \end{array}$$

- (8) The quotient of a lattice by a sub-lattice is an $(n-1)$ -Tate object.
- (9) The Sato Grassmannian of an elementary n -Tate object is a directed poset.

Remark 7.4. The observations behind remarks 3.32 and 5.26 similarly show that

$$K_i(\mathbf{n}\text{-Tate}_{\kappa}(\mathcal{C})) \cong K_{i-n}(\mathcal{C})$$

when \mathcal{C} is idempotent complete.

Proposition 7.5. For any X in an exact category \mathcal{C} , denote by $X[t]$, $X[[t]]$ and $X((t))$ the admissible Ind, Pro and elementary Tate objects

$$\begin{aligned} X[t] &:= \text{colim}_{n \in \mathbb{N}} X^{\oplus n}, \\ X[[t]] &:= \lim_{n \in \mathbb{N}} X^{\times n}, \text{ and} \\ X((t)) &:= X[[t]] \oplus X[t] \in \text{Tate}_{\mathbb{N}_0}^{el}(\mathcal{C}). \end{aligned}$$

Similarly, we define $X((t_1)) \cdots ((t_n)) \in \mathbf{n}\text{-Tate}_{\mathbb{N}_0}^{el}(\mathcal{C})$ by

$$X((t_1)) \cdots ((t_n)) := (X((t_1)) \cdots ((t_{n-1})))((t_n))$$

Now suppose that \mathcal{C} is a split exact category for which there exists a collection of objects $\{S_i\}_{i \in \mathbb{N}} \subset \mathcal{C}$ such that every object $Y \in \mathcal{C}$ is a direct summand of $\bigoplus_{i=0}^n S_i$ for some n . As above, denote by $\widehat{\prod_{\mathbb{N}} S}$ and $\widehat{\bigoplus_{\mathbb{N}} S}$ the admissible Pro and Ind-objects

$$\begin{aligned} \widehat{\prod_{\mathbb{N}} S} &:= \lim_{(m,n) \in \mathbb{N} \times \mathbb{N}} \prod_{i=0}^m S_i^{\times n}, \text{ and} \\ \widehat{\bigoplus_{\mathbb{N}} S} &:= \text{colim}_{(m,n) \in \mathbb{N} \times \mathbb{N}} \bigoplus_{i=0}^m S_i^{\oplus n}. \end{aligned}$$

Then every countable n -Tate object in \mathcal{C} is a direct summand of

$$\left(\widehat{\prod_{\mathbb{N}} S} \oplus \widehat{\bigoplus_{\mathbb{N}} S} \right) ((t_2)) \cdots ((t_n)).$$

Proof. We induct on n . Proposition 5.18 establishes the case $n = 1$. Assume the result is true for n . Then the category $\mathbf{n}\text{-Tate}_{\mathbb{N}_0}(\mathcal{C})$ is split exact and every object is a direct summand of $(\widehat{\prod_{\mathbb{N}} S} \oplus \widehat{\bigoplus_{\mathbb{N}} S})((t_2)) \cdots ((t_n))$. We can therefore apply Proposition 5.18 to $\mathbf{n}\text{-Tate}_{\mathbb{N}_0}(\mathcal{C})$ and conclude that every object in $\text{Tate}^{el}(\mathbf{n}\text{-Tate}_{\mathbb{N}_0}(\mathcal{C}))$ is a direct summand of

$$(\widehat{\prod_{\mathbb{N}} S} \oplus \widehat{\bigoplus_{\mathbb{N}} S})((t_2)) \cdots ((t_n))((t_{n+1})).$$

This completes the induction. \square

Example 7.6. Let R be a ring. Every countable n -Tate R -module is a direct summand of $R((t_1)) \cdots ((t_n))$.

7.2. Beilinson–Parshin Adèles.

Definition 7.7 (Beilinson–Parshin). Let X be an n -dimensional Noetherian scheme. For $0 \leq i \leq n$, denote by $|X|_i$ the set of points $p \in X$ such that the closure of p is an i -dimensional sub-scheme of X . Given $p \in X$, denote the inclusion of its closure by $j_{\overline{p}}: \overline{p} \hookrightarrow X$. Denote the inclusion of the r^{th} -order formal neighborhood of its closure by $j_{\overline{p}^r, *}: \overline{p}^r \hookrightarrow X$. Define the n -dimensional adèles

$$\text{QCoh}(X) \xrightarrow{\mathbb{A}_X^n(-)} \text{Mod}(\mathcal{O}_X)$$

to be the functor which commutes with direct limits and whose restriction to $\text{Coh}(X)$ is inductively given by,

- (1) for $n = 0$, $\mathbb{A}_X^0(\mathcal{F}) := \mathcal{F}$, and
- (2) for $n > 0$,
 - (a) if X is irreducible with generic point η , define

$$\mathbb{A}_X^n(\mathcal{F}) := \text{colim}_i \prod_{p \in |X|_{n-1}} \lim_r j_{\overline{p}^r, *} \mathbb{A}_{\overline{p}^r}^{n-1}(j_{\overline{p}^r}^* \mathcal{F}_i)$$

where the colimit is over the poset of coherent subsheaves $\mathcal{F}_i \subset j_{\eta, *} j_{\eta}^* \mathcal{F}$ such that $j_{\eta}^* \mathcal{F}_i = j_{\eta}^* \mathcal{F}$.

- (b) if X has irreducible components $\{X_a\}$, then

$$\mathbb{A}_X^n(\mathcal{F}) := \bigoplus_a j_{X_a, *} \mathbb{A}_{X_a}^n(j_{X_a}^* \mathcal{F}).$$

Remark 7.8. This definition corresponds to the *reduced* n -dimensional adèles of [13].

Example 7.9. When $X = \text{Spec}(\mathbb{Z})$, $\mathbb{A}_X^1(\mathcal{O}_X)$ is the finite adèles $\mathbb{Q} \otimes (\prod_p \mathbb{Z}_p)$.

We can also discuss the adèles at a single place.

Definition 7.10. Let X be an n -dimensional Noetherian scheme. Let $\xi := (p_0 < \dots < p_n)$ be an increasing sequence of points in X , with $\overline{\{p_i\}}$ of dimension i . Denote by $d(\xi)$ the sequence $d(\xi) := (p_0 < \dots < p_{n-1})$. Using the notation of Definition 7.7, we define the n -dimensional adèles at the place ξ

$$\text{QCoh}(X) \xrightarrow{\mathbb{A}_{X, \xi}^n(-)} \text{Mod}(\mathcal{O}_X)$$

to be the functor which commutes with direct limits and whose restriction to $\text{Coh}(X)$ is inductively given by,

- (1) for $n = 0$, $\mathbb{A}_{X,\xi}^0(\mathcal{F}) := \mathcal{F}$, and
- (2) for $n > 0$, define

$$\mathbb{A}_{X,\xi}^n(\mathcal{F}) := \operatorname{colim}_i \lim_r j_{p_{n-1},*} \mathbb{A}_{p_{n-1},d(\xi)}^{n-1}(j_{p_{n-1}}^* \mathcal{F}_i)$$

where the colimit is over the poset of coherent subsheaves $\mathcal{F}_i \subset j_{p_n,*} j_{p_n}^* \mathcal{F}$ such that $j_{p_n}^* \mathcal{F}_i = j_{p_n}^* \mathcal{F}$.¹²

Beilinson [3] formulated the n -dimensional adèles as the top degree piece of a functorial flasque resolution $\mathcal{F} \rightarrow \mathbb{A}_X^\bullet(\mathcal{F})$ of a coherent sheaf \mathcal{F} .¹³ In particular, $\mathbb{A}_X^n(-)$ is an exact functor with a canonical natural surjection

$$\mathbb{A}_X^n(-) \rightarrow H^n(X; -).$$

Following Parshin, Beilinson used this to express the Grothendieck trace map via a sum of residues, in analogy with Tate's work [29] on curves.

Denote by $\operatorname{Coh}_0(X) \subset \operatorname{Coh}(X)$ the full sub-category consisting of sheaves with 0-dimensional support, and define $0 - \operatorname{Tate}^{el}(\operatorname{Coh}_0(X)) := \operatorname{Coh}_0(X)$.

Theorem 7.11. *Let X be an n -dimensional Noetherian scheme.*

- (1) *The n -dimensional adèles give an exact functor*

$$\operatorname{Coh}(X) \xrightarrow{\mathbb{A}_X^n(-)} n\text{-}\operatorname{Tate}^{el}(\operatorname{Coh}_0(X)).$$

- (2) *Let $\xi := (p_0 < \dots < p_n)$ be an increasing sequence of points in X , with $\overline{\{p_i\}}$ of dimension i . The n -dimensional adèles at the place ξ give an exact functor*

$$\operatorname{Coh}(X) \xrightarrow{\mathbb{A}_{X,\xi}^n(-)} n\text{-}\operatorname{Tate}^{el}(\operatorname{Coh}_0(X)).$$

Proof. We prove the result for the global adèles \mathbb{A}_X^n by induction on n ; the proof for $\mathbb{A}_{X,\xi}^n$ follows by similar reasoning.

For $n = 0$, there is nothing to show. Suppose that we have shown the result for $0 \leq m < n$.

From the definition, it is enough to prove the result for X irreducible. Let \mathcal{F} be a coherent sheaf on X . We use the inductive hypothesis to show that

$$\mathbb{A}_X^n(\mathcal{F}) \in \operatorname{Ind}^a(\operatorname{Pro}^a((n-1) - \operatorname{Tate}_\kappa^{el}(\operatorname{Coh}_{\leq 0}(X))).$$

We then conclude the proof by exhibiting a lattice of $\mathbb{A}_X^n(\mathcal{F})$.

Denote the generic point of X by η . By definition

$$\mathbb{A}_X^n(\mathcal{F}) := \operatorname{colim}_i \prod_{p \in |X|_{n-1}} \lim_r j_{p^r,*} \mathbb{A}_{p^r}^{n-1}(j_{p^r}^* \mathcal{F}_i)$$

where the colimit ranges over coherent subsheaves $\mathcal{F}_i \subset j_{\eta,*} j_{\eta}^* \mathcal{F}$ such that $j_{\eta}^* \mathcal{F}_i = j_{\eta}^* \mathcal{F}$. Our inductive hypothesis states that

$$\mathbb{A}_{p^r}^{n-1}(j_{p^r}^* \mathcal{F}_i) \in (n-1) - \operatorname{Tate}^{el}(\operatorname{Coh}_0(\overline{p^r})),$$

so

$$j_{p^r,*} \mathbb{A}_{p^r}^{n-1}(j_{p^r}^* \mathcal{F}_i) \in (n-1) - \operatorname{Tate}^{el}(\operatorname{Coh}_0(X)),$$

¹²Note that dimension considerations imply that p_n is the generic point of the irreducible component of X which contains ξ .

¹³For a detailed description of the full adèlic resolution, see Huber [13].

and we see that

$$\mathbb{A}_X^n(\mathcal{F}) \in \text{Ind}^a(\text{Pro}^a((n-1) - \text{Tate}_\kappa^{el}(\text{Coh}_0(X))).$$

To show that $\mathbb{A}_X^n(\mathcal{F})$ is an n -Tate object, it suffices to produce a lattice (Theorem 5.4). Denote by I the directed poset indexing coherent subsheaves $\mathcal{F}_i \subset j_{\eta,*}j_{\eta}^*\mathcal{F}$ such that $j_{\eta}^*\mathcal{F}_i = j_{\eta}^*\mathcal{F}$. For any $i \in I$, denote by I_i the final subset of I consisting of all \mathcal{F}_j containing \mathcal{F}_i . Note that $\text{colim}_{j \in I_i} \mathcal{F}_j \cong j_{\eta,*}j_{\eta}^*\mathcal{F}$.

We claim that the inclusion

$$\left(\prod_{p \in |X|_{n-1}} \lim_r j_{\overline{p^r},*} \mathbb{A}_{\overline{p^r}}^{n-1}(j_{\overline{p^r}}^* \mathcal{F}_i) \right) \hookrightarrow \mathbb{A}_X^n(\mathcal{F})$$

is a lattice. Our inductive hypothesis guarantees that

$$\prod_{p \in |X|_{n-1}} \lim_r j_{\overline{p^r},*} \mathbb{A}_{\overline{p^r}}^{n-1}(j_{\overline{p^r}}^* \mathcal{F}_i) \in \text{Pro}^a((n-1) - \text{Tate}^{el}(\text{Coh}_0(X))).$$

For each $j \in I_i$, we have a short exact sequence

$$0 \longrightarrow \mathcal{F}_i \longrightarrow \mathcal{F}_j \longrightarrow Q_j \longrightarrow 0$$

Our inductive hypothesis ensures that for each $p \in |X|_{n-1}$ and each r , we have an exact sequence

$$0 \longrightarrow j_{\overline{p^r},*} \mathbb{A}_{\overline{p^r}}^{n-1}(j_{\overline{p^r}}^* \mathcal{F}_i) \longrightarrow j_{\overline{p^r},*} \mathbb{A}_{\overline{p^r}}^{n-1}(j_{\overline{p^r}}^* \mathcal{F}_j) \longrightarrow j_{\overline{p^r},*} \mathbb{A}_{\overline{p^r}}^{n-1}(j_{\overline{p^r}}^* Q_j) \longrightarrow 0.$$

These exact sequences fit into an admissible Pro-diagram of exact sequences, and thus an exact sequence of admissible Pro-objects

$$\begin{aligned} 0 \longrightarrow \prod_{p \in |X|_{n-1}} \lim_r j_{\overline{p^r},*} \mathbb{A}_{\overline{p^r}}^{n-1}(j_{\overline{p^r}}^* \mathcal{F}_i) &\longrightarrow \prod_{p \in |X|_{n-1}} \lim_r j_{\overline{p^r},*} \mathbb{A}_{\overline{p^r}}^{n-1}(j_{\overline{p^r}}^* \mathcal{F}_j) \longrightarrow \\ &\longrightarrow \prod_{p \in |X|_{n-1}} \lim_r j_{\overline{p^r},*} \mathbb{A}_{\overline{p^r}}^{n-1}(j_{\overline{p^r}}^* Q_j) \longrightarrow 0 \end{aligned} .$$

We claim that

$$\prod_{p \in |X|_{n-1}} \lim_r j_{\overline{p^r},*} \mathbb{A}_{\overline{p^r}}^{n-1}(j_{\overline{p^r}}^* Q_j) \in (n-1) - \text{Tate}^{el}(\text{Coh}_0(X)).$$

Indeed, by definition, $j_{\eta}^*\mathcal{F}_i \cong j_{\eta}^*\mathcal{F}_j$, so the support of Q_j has dimension at most $n-1$. Since X is Noetherian, the support of Q_j is Noetherian as well. The support therefore contains finitely many irreducible components. In particular, $j_{\overline{p^r}}^*Q_j = 0$ for all but finitely many $p \in |X|_{n-1}$. Further, because Q_j is coherent, for each p such that Q_j is non-zero on \overline{p} , there exists $r < \infty$ such that $j_{\overline{p^s}}^*Q_j \cong j_{\overline{p^r}}^*Q_j$ for all $s \geq r$. We conclude that

$$\prod_{p \in |X|_{n-1}} \lim_r j_{\overline{p^r},*} \mathbb{A}_{\overline{p^r}}^{n-1}(j_{\overline{p^r}}^* Q_j) \in (n-1) - \text{Tate}^{el}(\text{Coh}_0(X))$$

is a finite direct sum of $(n-1)$ -Tate objects, so is an $(n-1)$ -Tate object itself.

We now take the colimit over I_i of the short exact sequences (7.1) to obtain a short exact sequence

$$\begin{aligned} 0 \longrightarrow \prod_{p \in |X|_{n-1}} \lim_r j_{\overline{p^r},*} \mathbb{A}_{\overline{p^r}}^{n-1}(j_{\overline{p^r}}^* \mathcal{F}_i) &\longrightarrow \mathbb{A}_X^n(\mathcal{F}) \longrightarrow \\ &\longrightarrow \text{colim}_{I_i} \prod_{p \in |X|_{n-1}} \lim_r j_{\overline{p^r},*} \mathbb{A}_{\overline{p^r}}^{n-1}(j_{\overline{p^r}}^* Q_j) \longrightarrow 0 \end{aligned} .$$

By construction,

$$\prod_{p \in |X|_{n-1}} \lim_r j_{p^r, *} \mathbb{A}_{p^r}^{n-1} (j_{p^r}^* \mathcal{F}_i) \in \text{Pro}^a((n-1) - \text{Tate}^{el}(\text{Coh}_0(X)))$$

while

$$\text{colim}_{I_i} \prod_{p \in |X|_{n-1}} \lim_r j_{p^r, *} \mathbb{A}_{p^r}^{n-1} (j_{p^r}^* Q_j) \in \text{Ind}^a((n-1) - \text{Tate}^{el}(\text{Coh}_0(X))).$$

We conclude that $\mathbb{A}_X^n(\mathcal{F}) \in \mathbf{n}\text{-Tate}^{el}(\text{Coh}_0(X))$. \square

APPENDIX A. REMARKS ON THE LEFT S-FILTERING CONDITION

In this appendix, we recall Schlichting's definition of left s-filtering [27], and we reproduce a proof, due to T. Bühler, that this definition is equivalent to Definition 2.16.

Definition A.1 (Schlichting). Let \mathcal{D} be an exact category. An exact, full subcategory $\mathcal{C} \subset \mathcal{D}$ is *Schlichting left s-filtering* if

- (1) for any exact sequence $X \hookrightarrow Y \twoheadrightarrow Z$ in \mathcal{D} , X and Z are in \mathcal{C} if and only if Y is,
- (2) \mathcal{C} is left filtering in \mathcal{D} (in the sense of Definition 2.15), and
- (3) for every admissible epic $F \twoheadrightarrow X$ in \mathcal{D} , with $X \in \mathcal{C}$, there exists an admissible monic $Y \hookrightarrow F$, with $Y \in \mathcal{C}$, such that the composite map $Y \twoheadrightarrow X$ is an admissible epic in \mathcal{C} .

Proposition A.2 (Bühler). *Definitions 2.16 and A.1 are equivalent, i.e. an exact, full sub-category $\mathcal{C} \subset \mathcal{D}$ satisfies Definition A.1 if and only if \mathcal{C} is left filtering and left special in \mathcal{D} .*

Proof. For the “only if”, it suffices to show that $\mathcal{C} \subset \mathcal{D}$ is left special if it satisfies the third condition of Definition A.1. By assumption, given an admissible epic $F \twoheadrightarrow X$ in \mathcal{D} with $X \in \mathcal{C}$, there exists a commuting square

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \parallel \\ F & \longrightarrow & X \end{array}$$

with $Y \twoheadrightarrow X$ an admissible epic in \mathcal{C} . The universal property of kernels implies that this square extends to a commuting diagram

$$\begin{array}{ccccc} Z & \hookrightarrow & Y & \longrightarrow & X \\ \downarrow & & \downarrow & & \parallel \\ G & \hookrightarrow & F & \longrightarrow & X \end{array}$$

in which the top row is an exact sequence in \mathcal{C} , while the bottom row is an exact sequence in \mathcal{D} . We conclude that \mathcal{C} is left special in \mathcal{D} .

For the “if”, suppose $\mathcal{C} \subset \mathcal{D}$ is left filtering and left special. We begin by showing that this implies that for every admissible epic $F \twoheadrightarrow X$ in \mathcal{D} with $X \in \mathcal{C}$, there exists a commuting diagram

$$\begin{array}{ccccc} Z & \hookrightarrow & Y & \twoheadrightarrow & X \\ \downarrow & & \downarrow & & \parallel \\ G & \hookrightarrow & F & \twoheadrightarrow & X \end{array}$$

in which the top row is an exact sequence in \mathcal{C} , the bottom row is an exact sequence in \mathcal{D} , and the vertical maps are admissible monics in \mathcal{D} ; note that if such a diagram always exists, then \mathcal{C} satisfies the third condition of Definition A.1, because we can take Y to be as in the diagram above.

To see that the diagram above exists, observe that, because \mathcal{C} is left special, there exists a diagram

$$\begin{array}{ccccc} Z & \hookrightarrow & Y & \twoheadrightarrow & X \\ \downarrow & & \downarrow & & \parallel \\ G & \hookrightarrow & F & \twoheadrightarrow & X \end{array}$$

with no assumptions on the vertical maps, in which the top row is exact in \mathcal{C} and the bottom row is exact in \mathcal{D} . Because \mathcal{C} is left filtering in \mathcal{D} , the map $Z \rightarrow G$ factors through an admissible monic $Z' \hookrightarrow G$ with $Z' \in \mathcal{C}$. Because pushouts of admissible monics along arbitrary maps exist and are admissible monics, we denote $Y' := Z' \cup_Z Y$ and observe that we have a commuting diagram

$$\begin{array}{ccccc} Z & \hookrightarrow & Y & \twoheadrightarrow & X \\ \downarrow & & \downarrow & & \parallel \\ Z' & \hookrightarrow & Y' & \twoheadrightarrow & X \\ \downarrow & & \downarrow & & \parallel \\ G & \hookrightarrow & F & \twoheadrightarrow & X \end{array} .$$

Because the top sequence is exact, and the upper left square is a pushout in which the horizontal maps are admissible monics, the middle row is exact by [9, Proposition 2.12]. Because the lower left vertical map is an admissible monic (by assumption), the 5-lemma [9, Corollary 3.2] implies that lower middle vertical map is also an admissible monic [9, Corollary 3.2]. We conclude that the bottom rectangle of this diagram is of the desired form.

It remains to show that if $\mathcal{C} \subset \mathcal{D}$ is left filtering and left special, then the first condition in Definition A.1 is satisfied. Let

$$X \hookrightarrow Y \twoheadrightarrow Z$$

be a short exact sequence in \mathcal{D} . Lemma 2.14 shows that Y is in \mathcal{C} if X and Z are. Conversely, suppose $Y \in \mathcal{C}$. Because \mathcal{C} is left filtering in \mathcal{D} , there exists a

commuting triangle

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & Z \\ & \searrow & \uparrow \\ & & W \end{array}$$

with W in \mathcal{C} . The commuting triangle implies that the map $W \longrightarrow Z$ is an epic admissible monic, i.e. an isomorphism. This shows that $Z \in \mathcal{C}$.

To show that X is in \mathcal{C} , we observe that there exists a commuting diagram

$$\begin{array}{ccccc} W & \hookrightarrow & Y' & \twoheadrightarrow & Z \\ \downarrow & & \downarrow & & \parallel \\ X & \hookrightarrow & Y & \twoheadrightarrow & Z \end{array}$$

in which the vertical maps are admissible monics in \mathcal{D} , and the top row is an exact sequence in \mathcal{C} . By [9, Proposition 2.12], the left hand square is a pushout, and

$$\operatorname{coker}(W \hookrightarrow X) \cong \operatorname{coker}(Y' \hookrightarrow Y).$$

We showed above that $\operatorname{coker}(Y' \hookrightarrow Y)$ is in \mathcal{C} . Because \mathcal{C} is closed under extensions (Lemma 2.14), we conclude that X is in \mathcal{C} as well. \square

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